EXPANSION PROBLEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH AUXILIARY CONDITIONS AT MORE THAN TWO POINTS*

BY

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1. Introduction

The boundary value and expansion problems for the equation of the nth order with boundary conditions at two points have been studied by Birkhoff.† Bôcher‡ has suggested the generalization of these results to the equation with auxiliary conditions at more than two points. Such generalization of the essential properties of the differential system has been carried out by the author, and in this paper is given the proof of the convergence of the expansion, which may be studied quite independently of the other results. The formal development of the boundary problem and a more detailed discussion of the form of the series will be presented in other papers.

The differential equation is taken in the form

$$\frac{d^n u}{dx^n} + * + p_2 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + p_n u + \lambda u = 0,$$

in which $p_2, p_3, \cdots, p_n$, are functions of the real variable x, continuous together with their derivatives of all orders in the closed interval (a, b). The auxiliary conditions apply to k points of the interval (a, b), of which the first and the last are the ends of the interval. These points, arranged in order of ascending algebraic magnitude, will be denoted by

$$a_1 = a, a_2, a_3, \cdots, a_{k-1}, a_k = b.$$

* Presented to the Society, April 29, 1916.

† Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, these Transactions, vol. 9 (1908), pp. 373–395. See also Tamarkine, *Sur quelques points de la théorie des équations différentielles linéaires ordinaires et sur la généralisation de la série de Fourier*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 345–382; and Birkhoff, *Note on the expansion problems of ordinary linear differential equations*, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 115–126.

‡ Bôcher, *Boundary problems in one dimension*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1913,vol. 1, pp. 163–195; see p. 166.

The auxiliary conditions may then be written

$$W_i(u) \equiv W_{1i}(u) + W_{2i}(u) + \cdots + W_{ki}(u) = 0 \quad (i = 1, 2, \cdots, n),$$

in which

$$W_{ji}(u) \equiv \alpha_{ji}\, u(a_j) + \alpha'_{ji}\, u'(a_j) + \alpha''_{ji}\, u''(a_j) + \cdots + \alpha_{ji}^{(n-1)}\, u^{(n-1)}(a_j),$$

and $\alpha_{ji}, \alpha'_{ji}, \alpha''_{ji}, \cdots, \alpha_{ji}^{(n-1)}$, are constants, real or complex.

It is possible to define explicitly in terms of the auxiliary conditions and a fundamental system of solutions of the differential equation, a function $G(x, s; \lambda)$, having the properties of a Green's function of the system. These properties will be discussed elsewhere with the exception of a single one, which forms the subject of this paper, namely that

$$f(x) = \lim_{r=\infty} \frac{1}{2\pi i} \int_\Gamma \int_a^b G(x, s; \lambda) f(s)\, ds\, d\lambda, \tag{1}$$

where Γ is a contour in the λ-plane enclosing the first r poles of $G(x, s; \lambda)$, and $f(x)$ satisfies certain conditions of continuity, in some cases of the sort usually imposed, in others more restrictive, according to the nature of the auxiliary conditions. The auxiliary conditions themselves are subject to restrictions beyond those found necessary in the two point case. This integral may be regarded as the sum of the first terms of an infinite series of residues, corresponding to the series of orthogonal functions in the familiar cases, and these residues can in many cases be expressed linearly in terms of the characteristic functions of the differential system. Since the integral will be considered here on its own merits this interpretation will not be considered further.

2. Preliminary formulas

A fundamental system of solutions of the differential equation may be chosen analytic in λ. Let $y_1(x, \lambda), y_2(x, \lambda), \cdots, y_n(x, \lambda)$, be such a system. Then let $D(\lambda)$ be the determinant which has $W_i(y_j)$ for the element in the ith row and jth column. Its roots are the characteristic numbers for the system, but will concern us only as points where an expression having $D(\lambda)$ in its denominator ceases to be analytic. Let $D_1(s, \lambda)$ be the determinant which has the $(n - i)$th derivative of $y_j(s)$ for the element in the ith row and jth column, and let $D_2(x, s; \lambda)$ be this determinant with the elements in the first row replaced by the $y_j(x)$ with corresponding subscript. Then let

$$\bar{G}(x, s; \lambda) = \pm \tfrac{1}{2} D_2/D_1, \qquad + \text{if } x > s, \qquad - \text{if } x < s. \tag{2}$$

If now $N(x, s; \lambda)$ is the determinant formed by bounding $D(\lambda)$ on the top by the $y_j(x)$ with corresponding subscript, and on the right by the $W_i(\bar{G})$

with corresponding subscript, and filling in the upper right-hand corner with $\bar{G}(x, s; \lambda)$, then the function $G(x, s; \lambda)$ mentioned above is given by the formula

$$G(x, s; \lambda) = (-1)^n N/D. \tag{3}$$

The proof of the convergence of the integral is based on the use of the following asymptotic solutions of the differential equation, due to Birkhoff.* Make the substitution $\lambda = \rho^n$. Then on any region S of the ρ-plane, defined by the relation

$$\frac{l\pi}{n} \leqq \arg \rho \leqq \frac{(l+1)\pi}{n},$$

in which l is a positive integer less than $n > 1$, there exist n linearly independent solutions, analytic in ρ, and such that

$$\begin{aligned} y_i(x, \rho) &= u_i(x, \rho) + e^{\rho w_i(x-a)} \frac{E_1}{\rho^m}, \\ \frac{dy_i}{dx} &= \frac{du_i}{dx} + e^{\rho w_i(x-a)} \frac{E_2}{\rho^{m-1}}, \qquad (i = 1, 2, \cdots, n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \frac{d^{n-1} y_i}{dx^{n-1}} &= \frac{d^{n-1} u_i}{dx^{n-1}} + e^{\rho w_i(x-a)} \frac{E_n}{\rho^{m-n+1}}, \end{aligned} \tag{4}$$

where

$$u_i(x, \rho) = e^{\rho w_i(x-a)} \left\{ 1 + \frac{u_{i1}(x)}{\rho} + \cdots + \frac{u_{im}(x)}{\rho^m} \right\}.$$

Here m is any positive integer, the w's are the roots of the equation

$$w^n + 1 = 0,$$

the functions $u_{ij}(x)$ are finite and continuous for all values of x in the interval (a, b), and E is a generic notation for functions of ρ (and other variables) uniformly bounded for large values of ρ. In part of the work it will be necessary to use a more explicit form of the functions $u_{ij}(x)$, due to Milne,† viz., $u_{ij}(x) = \phi_j(x)/(w_i)^j$. The functions ϕ are then the same for all the n solutions $y_i(x, \lambda)$.

By the substitution $\lambda = \rho^n$ the λ-plane goes over into two adjacent S regions, S_1 and S_2. The integral (1) becomes

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\gamma_1} \int_a^b n\rho^{n-1} G(x, s; \rho^n) f(s)\, ds d\rho \\ &\qquad + \frac{1}{2\pi i} \int_{\gamma_2} \int_a^b n\rho^{n-1} G(x, s; \rho^n) f(s)\, ds d\rho, \end{aligned} \tag{5}$$

* Birkhoff, loc. cit., p. 381.

† Milne, *Note on asymptotic expressions in the theory of linear differential equations*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 166–169.

in which γ_1 and γ_2 may be taken as arcs of the same circle with center at the origin.

Before substituting these solutions in the auxiliary conditions, the conditions are normalized as follows:

First, the number of conditions of order $n-1$ at the points a and b is reduced, by means of linear combinations, to two at most; then, the remaining conditions of order $n-2$ at these points are reduced to two at most; and so on.

It is convenient to introduce Birkhoff's notation, by which $[w]$ represents an expression of the form

$$w+\frac{a}{\rho}+\frac{b}{\rho^2}+\frac{c}{\rho^3}+\cdots+\frac{E}{\rho^m},$$

in which w, a, b, etc., are independent of ρ but need not be constants, and E is a function of ρ and other variables bounded for large values of ρ.

The result of substituting the asymptotic solutions into the auxiliary conditions is

$$\begin{aligned}
W_{1i}(y_j) &= (\rho w_j)^{h_{1i}}[\alpha_{1i}], \\
W_{2i}(y_j) &= (\rho w_j)^{h_{2i}}[\alpha_{2i}]\, e^{\rho w_j(a_2-a)}, \\
&\cdots\cdots\cdots\cdots, \\
W_{ki}(y_j) &= (\rho w_j)^{h_{ki}}[\alpha_{ki}]\, e^{\rho w_j(b-a)}.
\end{aligned}$$

Because of the normalization of the auxiliary conditions $h_{1i}=h_{ki}\leqq h_{ji}$, $j=2,3,\cdots,k-1$. So let us set $h_{1i}=h_{ki}=h_i$, and we have $h_{i-1}\geqq h_i \geqq h_{i+1}$, and no three successive h_i's are equal.

In the odd order case let $n=2\mu-1$. Then the indices 1 to n can be determined in such a way that for any ρ in S

$$R(\rho w_1)\leqq R(\rho w_2)\leqq R(\rho w_3)\leqq\cdots\leqq R(\rho w_n),$$

where $R(w)$ indicates the real part of w. Hence for S

$$\begin{aligned}
R(\rho w_i) &< 0 && (i=1,2,\cdots,\mu-1), \\
&> 0 && (i=\mu+1,\mu+2,\cdots,n), \\
&= 0 && \left(i=\mu \text{ and } \arg\rho=\frac{l_0+\frac{1}{2}}{n}\pi\right),
\end{aligned}$$

and it follows that

$$\begin{aligned}
W_i(y_j) &= (\rho w_j)^{h_i}[\alpha_{1i}] && (j=1,2,\cdots,\mu-1), \\
&= (\rho w_j)^{h_i}[\alpha_{ki}]\, e^{\rho w_j(b-a)} && (j=\mu+1,\mu+2,\cdots,n), \\
&= (\rho w_j)^{h_i}[\alpha_{1i}]+(\rho w_j)^{h_i}[\alpha_{ki}]\, e^{\rho w_j(b-a)}+\sum_{l=2}^{k-1}(\rho w_j)^{h_{li}}[\alpha_{li}]\, e^{\rho w_j(a_l-a)} \\
& && (j=\mu).
\end{aligned}$$

In the even order case let $n = 2\mu$. Then the indices 1 to n can be determined in such a way that for any ρ in S

$$\begin{aligned} R(\rho w_i) &< 0 && (i = 1, 2, 3, \cdots, \mu - 1), \\ &> 0 && (i = \mu + 2, \cdots, n), \\ &= 0 && \left(i = \mu, \mu + 1, \text{ and either } \arg \rho = \frac{l_0 \pi}{n} \text{ or } \arg \rho = \frac{l_0 + 1}{n} \pi\right), \end{aligned}$$

and since $-w_\mu = w_{\mu+1}$, we have

$$\begin{aligned} W_i(y_j) &= (\rho w_j)^{h_i}[\alpha_{1i}] && (j = 1, 2, \cdots, \mu - 1), \\ &= (\rho w_j)^{h_i}[\alpha_{ki}]\, e^{\rho w_j (b-a)} && (j = \mu + 2, \cdots, n), \\ &= (\rho w_\mu)^{h_i}[\alpha_{1i}] + (\rho w_\mu)^{h_i}[\alpha_{ki}]\, e^{\rho w_\mu (b-a)} + \sum_{l=2}^{k-1} (\rho w_\mu)^{h_l}[\alpha_{li}]\, e^{\rho w_\mu (a_l - a)} && (j = \mu), \\ &= (-\rho w_\mu)^{h_i}[\alpha_{1i}] + (-\rho w_\mu)^{h_i}[\alpha_{ki}]\, e^{-\rho w_\mu (b-a)} \\ &\qquad + \sum_{l=2}^{k-1} (-\rho w_\mu)^{h_l}[\alpha_{li}]\, e^{-\rho w_\mu (a_l - a)} && (j = \mu + 1). \end{aligned}$$

3. The function $D(\lambda)$

Preparatory to taking up the main convergence problem it is necessary to study the distribution of those points where the function $D(\lambda)$ vanishes or becomes nearly equal to zero. This study is simplified by considering separately the cases where n is odd and where n is even.

Odd order case. When n is odd, the substitution of the asymptotic expressions for the W's in the determinant $D(\lambda)$ gives

$$D(\lambda) = \prod_{i=1}^{n} \rho^{h_i} \prod_{j=\mu+1}^{n} e^{\rho w_j (b-a)}\, \Delta,$$

in which Δ is that determinant which has for the element in the ith row and jth column, $[\alpha_{1i}]\, w_j^{h_i}$ for $j = 1, 2, \cdots, \mu - 1$; $[\alpha_{ki}]\, w_j^{h_i}$ for $j = \mu + 1, \mu + 2, \cdots, n$, and finally

$$B_i \equiv w_\mu^{h_i}([\alpha_{1i}] + [\alpha_{ki}]\, e^{\rho w_\mu (b-a)} + \sum_{l=2}^{k-1} (\rho w_\mu)^{h_u - h_i}[\alpha_{li}]\, e^{\rho w_\mu (a_l - a)}),$$

for $j = \mu$. Let ν be the highest power to which ρw_μ enters in any B_i; then Δ can be expanded in the form

$$\Delta \equiv [\theta_1] + [\theta_k]\, e^{\rho w_\mu (b-a)} + (\rho w_\mu)^\nu \left\{ \sum_{l=2}^{k-1} [\theta_l]\, e^{\rho w_\mu (a_l - a)} \right\}. \tag{6}$$

We shall now prove the

Theorem. *If in* (6) *neither* θ_1 *nor* θ_k *is zero, and if for* $\nu \geqq 1$ *neither* θ_2 *nor* θ_{k-1} *is zero, then* $D(\lambda)$ *has an infinite number of roots, and there exist two absolute constants* ω *and* Ω *such that the number of roots of* $D(\lambda)$ *in any circle*

about the origin of radius r is less than Ωr and greater than ωr and, finally, for values of λ uniformly away from the roots, $D(\lambda)$ remains uniformly away from zero.

This last statement is made more explicit by inequalities (16) and (21) below.

We assume, then, that neither θ_1 nor θ_k is zero. The number ν may be negative, zero, or positive. If it is negative Δ can be written more simply

$$[\theta_1] + [\theta_k]\, e^{\rho w_\mu (b-a)}.$$

Birkhoff* has shown that the roots of this are asymptotically equal to

$$\rho_l = \frac{2l\pi\sqrt{-1}}{w_\mu (b-a)} + \frac{\log\left(-\frac{\theta_1}{\theta_k}\right)}{w_\mu (b-a)},$$

and that Δ remains uniformly away from zero if ρ is uniformly away from a root.

When ν is zero, Δ becomes

$$[\theta_1] + [\theta_k]\, e^{\rho w_\mu (b-a)} + \sum_{l=2}^{k-1} [\theta_l]\, e^{\rho w_\mu (a_l - a)}, \tag{7}$$

and now at least one other θ in addition to θ_1 and θ_k is assumed to be different from zero. In treating this case the distribution of roots of the function obtained by suppressing the brackets is determined, and then the roots of Δ are proved asymptotically equal to these.

It is convenient to make a change of variable. Let $\rho w_\mu (a_2 - a) = z$, and divide through by θ_1. The equation to be considered is, then,

$$f(z) \equiv 1 + Q_0 e^{z} + Q_1 e^{c_1 z} + Q_2 e^{c_2 z} + \cdots + Q_{k-2} e^{c_{k-2} z} = 0. \tag{8}$$

The Q's are complex constants, and in particular Q_{k-2} and at least one other are different from zero. The c's are real constants such that $1 < c_1 < c_2 < \cdots < c_{k-2}$. Let $z = x + yi$, and let $R(\phi)$ denote the real part of ϕ, and $I(\phi)$ the coefficient of the pure imaginary part. Let ϵ be an arbitrary small positive quantity, and choose X so large that for $x = -X$, $f(z)$ differs from 1 by less than ϵ, and at the same time for $x = X$, $f(z)/e^{c_{k-2} z}$ differs from Q_{k-2} by less than ϵ. The region between $x = X$ and $x = -X$ will be called the region R throughout this paper. We shall suppose that ϵ is chosen small enough so that $f(z)$ does not vanish outside of R.

Now $f(z)$ is analytic in any finite region. Hence in any interval, no matter how small, along the y-axis, it is possible to find a line $y =$ constant along which $f(z)$ does not vanish. Let $y = Y_1$ and $y = Y_2$ be any two such lines. The number of roots of $f(z)$ in the rectangle $x = X, y = Y_1, x = -X, y = Y_2$ is determined by following the change in arc $f(z)$ as z describes this

* Birkhoff, loc. cit., pp. 383–5.

rectangle. Now

$$\operatorname{arc} f(z) = \sin^{-1}\frac{I(f)}{|f|},$$

and for y constant $I(f)$ has the form

$$I(f) \equiv d_0 e^{x} + d_1 e^{c_1 x} + d_2 e^{c_2 x} + \cdots + d_{k-2} e^{c_{k-2} x},$$

in which the d's are real constants. The finite roots of $I(f)$ are the same as those of

$$I_0(f) \equiv d_0 + d_1 e^{(c_1-1)x} + d_2 e^{(c_2-1)x} + \cdots + d_{k-2} e^{(c_{k-2}-1)x}.$$

But between any two roots of this is a root of the derivative

$$I_0'(f) \equiv d_1(c_1-1)e^{(c_1-1)x} + d_2(c_2-1)e^{(c_2-1)x} + \cdots + d_{k-2}(c_{k-2}-1)e^{(c_{k-2}-1)x},$$

while its roots are the same as those of

$$I_1(f) \equiv d_1(c_1-1) + d_2(c_2-1)e^{(c_2-c_1)x} + \cdots + d_{k-2}(c_{k-2}-1)e^{(c_{k-2}-c_1)x}.$$

The argument is repeated till finally I_{k-2} is reached, which has only one term and does not vanish for x finite. Thus $I(f)$ has at most $(k-2)$ finite roots. Should any of the I's have multiple roots the argument must be slightly changed, but the result is, of course, the same.

Now as z traces the line $x = -X$, $f(z)$ remains very nearly equal to 1, so that the resulting change in arc $f(z)$ is less than some small positive quantity η_1 which approaches zero with ϵ. As z traces the line $x = X$ the change in arc $f(z)$ differs but slightly from $c_{k-2}(Y_2 - Y_1)$, as is at once apparent from the formula

$$\operatorname{arc} f(z) = \operatorname{arc} e^{c_{k-2} z} + \operatorname{arc}\frac{f(z)}{e^{c_{k-2} z}}.$$

In other words, the change in arc along this line can be written

$$c_{k-2}(Y_2 - Y_1) \pm \eta_2,$$

where η_2 is a small positive quantity which approaches zero with ϵ. Finally along the lines $y = Y_1$ and $y = Y_2$ the function $\sin\{\operatorname{arc} f(z)\}$ vanishes at most $k-2$ times, whence arc $f(z)$ increases by less than $(k-1)\pi + \eta_3$ and $(k-1)\pi + \eta_4$ respectively, where η_3 and η_4 are again small positive quantities which approach zero with ϵ. To sum up, as z traces the rectangle the resulting change in arc $f(z)$ is not greater than

$$c_{k-2}(Y_2 - Y_1) + 2(k-1)\pi + \eta$$

nor less than

$$c_{k-2}(Y_2 - Y_1) - 2(k-1)\pi - \eta \quad (\eta = \eta_1 + \eta_2 + \eta_3 + \eta_4),$$

if $Y_2 - Y_1$ is large enough to make this expression positive. Since η can be

made arbitrarily small by taking ϵ small enough, it can be dropped from these inequalities. So finally in any interval of R of length l the number of roots of $f(z)$ is not greater than $c_{k-2}\, l/2\pi + k - 1$, nor less than $c_{k-2}\, l/2\pi - k + 1$, if this number is positive. In particular, by making l very small, one sees that a multiple root of $f(z)$ cannot be of order higher than $k - 1$.

It is now necessary to prove that if z remains uniformly away from a root of $f(z)$ then $f(z)$ remains uniformly away from zero. This is proved by means of the following auxiliary theorem:

THEOREM. *Given $f(z, x_1, x_2, \cdots, x_n)$, continuous in all of its arguments and analytic in z for $a_i \leqq x_i \leqq b_i$ and z in a closed finite region S of the complex plane, and such that for no set of values of the x's is the number of roots of $f(z)$ greater than a given constant N; if for any set of values of the x's, z is a point at a distance greater than δ from a zero of $f(z)$ and from the boundary of S, then*

$$|f(z, x_1, x_2, \cdots, x_n)| \geqq \lambda,$$

where λ is a real positive constant independent of the x's.

It is no restriction to assume that δ is small enough so that for each set of values of the x's the corresponding region of variation for z, defined by the hypotheses, actually contains points. This follows from the assumption that for any set of values of the x's the number of roots of $f(z)$ is less than N. Now for any particular set of values of the x's there is a λ, such that for these x's and for z restricted to its corresponding region of variation,

$$|f(z, x_1, x_2, \cdots, x_n)| \geqq \lambda,$$

while there is a value of z for which the equality holds. We have thus defined a function of the x's, $\lambda(x_1, x_2, \cdots, x_n)$, which, by means of the easily proved fact that the zeros of $f(z, x_1, x_2, \cdots, x_n)$ are continuous functions of the x's, can be shown to be continuous in the closed region $a_i \leqq x_i \leqq b_i$. Since $\lambda(x_1, x_2, \cdots, x_n)$ is continuous in a closed region, it has a lower limit which it attains. Call this limit λ. Then for some value of z and the x's, $|f| = \lambda$. Hence λ is not equal to zero, for f does not vanish in any of the regions for which the function of which λ is the lower limit was defined. Hence the theorem is proved.

First apply this theorem to the function

$$f(z, \phi_1, \phi_2, \cdots, \phi_{k-2}) \equiv 1 + Q_0 e^z + Q_1 e^{e^{1z} + \phi_1 i} + \cdots + Q_{k-2} e^{e^{k-2 z} + \phi_{k-2} i}.$$

The region S can be that portion of R for which $0 \leqq y \leqq 2\pi$, augmented by a strip of width δ surrounding it. Obviously the breadth of R, that is, the value of X, depends only on the Q's and not on the ϕ's. It is also clear that the function satisfies the conditions of the theorem. Let λ indicate the number determined by the theorem corresponding to δ.

Consider, then, the function (8) in R. Any interval of R,

$$2l\pi \leqq y \leqq 2(l+1)\pi,$$

can, by a transformation $\bar{y} = y + 2l\pi$ be brought to the interval $0 \leqq y \leqq 2\pi$, and f becomes

$$f(z + 2l\pi i) \equiv 1 + Q_0 e^z + Q_1 e^{c_1 z - 2l\pi c_1 i} + \cdots + Q_{k-2} e^{c_{k-2} z - 2l\pi c_{k-2} i}.$$

But, without changing f, each expression $2l\pi c_j i$ can be replaced by a $\phi_j i$ where $0 \leqq \phi \leqq 2\pi$. So $f(z + 2l\pi i)$ has the form above, and for z at a greater distance from a root than δ, $|f(z + 2l\pi)| \geqq \lambda$. By giving l all integral values, all of R is covered by these intervals, and so finally, for z in R and at a distance from a root of $f(z)$ greater than δ, $|f(z)| \geqq \lambda > 0$.

We can now prove that the roots of (7) are asymptotically equal to those of (8). First it is clear that by choosing ϵ small enough we can be sure that all the roots of (7) lie within the region R. Let us denote (7) by $\psi(\rho)$ and (8) by $f(\rho)$. Then in R

$$\psi(\rho) \equiv \theta_1 f(\rho) + \frac{E}{\rho},$$

from which it is obvious that the roots of $\psi(\rho)$ are very nearly equal to those of $f(\rho)$ for large values of ρ. Now

$$\psi(\rho) \equiv \theta_1 f(\rho)\left(1 + \frac{E}{\theta_1 f(\rho)\rho}\right), \tag{9}$$

whence

$$\operatorname{arc}\psi(\rho) \equiv \operatorname{arc}\theta_1 + \operatorname{arc} f(\rho) + \operatorname{arc}\left(1 + \frac{E}{\theta_1 f(\rho)\rho}\right).$$

Let δ be an arbitrary small positive quantity. Then according to the previously determined distribution of the roots of $f(\rho)$, it is possible to enclose these roots in groups of less than k in simply connected regions of major diameter less than $k\delta$, and such that no point of the boundary is at a distance from a root of $f(\rho)$ less than δ. On the boundaries of these regions we have then $|f(\rho)| \geqq \lambda$, where λ is determined by the theorem above. Hence for large values of ρ, the expression $E/(\theta_1 f(\rho)\rho)$ is very small on these boundaries, and so the resulting change in arc $(1 + E/(\theta_1 f(\rho)\rho))$ as z traces the boundary of a region is zero, and the change in arc $\psi(\rho)$ is the same as the change in arc $f(\rho)$. So the number of roots of $\psi(\rho)$ in any such region is for large values of ρ the same as the number of roots of $f(\rho)$ in it; that is, the roots of $\psi(\rho)$ are asymptotically equal to those of $f(\rho)$.

From this together with (9) can be deduced the fact that for large values of ρ in R the function $\psi(\rho)$ remains uniformly away from zero when ρ is uniformly away from a root of $\psi(\rho)$. Outside of R the function $\psi(\rho)$ of course remains uniformly away from zero.

It is convenient to consider next the case in which Δ reduces to

$$[\theta_1] + [\theta_k]\, e^{\rho w_\mu (b-a)} + (\rho w_\mu)^\nu\, [\theta_2]\, e^{\rho w_\mu (a_2 - a)},$$

which, in particular, it always does if $k = 3$. It is assumed that none of the θ's are zero, and that $\nu \geqq 1$. Then

$$\Delta \equiv [\theta_2]\, e^{\rho w_\mu (a_2 - a)}\, \Delta_0,$$

where

$$\Delta_0 \equiv (\rho w_\mu)^\nu + [\theta_b]\, e^{\rho w_\mu (b - a_2)} + [\theta_a]\, e^{-\rho w_\mu (a_2 - a)}, \tag{10}$$

and in this

$$[\theta_a] \equiv \frac{[\theta_1]}{[\theta_2]}, \qquad [\theta_b] \equiv \frac{[\theta_k]}{[\theta_2]}.$$

It is necessary to consider separately the part of S in which $R(\rho w_\mu)$ is positive and the part in which it is negative.

$R(\rho w_\mu) \leqq 0$. It will now be shown that in that part of S where $R(\rho w_\mu) \leqq 0$, the roots of Δ_0 are asymptotically equal to those of

$$f(\rho) \equiv (\rho w_\mu)^\nu + \theta_a\, e^{-\rho w_\mu (a_2 - a)}.$$

To find the roots of $f(\rho)$ set $\rho w_\mu = x + yi = re^{i\phi}$, and $-\theta_a = r_a e^{i\phi_a}$. When $f(\rho)$ is equated to zero the second term may be transposed, and then, when absolute values and angles are equated, the two following equations result:

$$r^\nu = r_a\, e^{x(a_2 - a)}, \qquad \nu\phi = \phi_a - y(a_2 - a) + 2m\pi \quad (m = 0, \pm 1, \pm 2, \pm 3, \cdots).$$

The zeros of $f(\rho)$ are the points of intersection of these two curves. Since, in that part of the ρ plane under consideration, $x \leqq 0$, $y > 0$, the first curve is equivalent to

$$y = \sqrt{r_a^{2/\nu}\, e^{-2x(a_2 - a)/\nu} - x^2},$$

and this has but one branch which is in character much like the exponential curve, and which for large values of ρ approaches parallelism to the line $R(\rho w_\mu) = 0$, i. e.,

$$\lim_{\rho = \infty} (dx/dy) = 0.$$

It is also easily shown that

$$\lim_{\rho = \infty} (x/y)^2 = 0,$$

which is to say that for points on this curve

$$\lim_{\rho = \infty} \phi = \pi/2 + 2l\pi \qquad (l = 0, \pm 1, \pm 2, \pm 3, \cdots).$$

When this is substituted in the second curve it is found that the roots of $f(\rho)$ are asymptotically equal to points whose y coördinates are given by

$$y = \frac{\phi_a + 2m\pi - \nu(\frac{1}{2}\pi + 2l\pi)}{a_2 - a},$$

and since ν is an integer this is equivalent to

$$y = \frac{\phi_a + 2m\pi - \frac{1}{2}\nu\pi}{a_2 - a}.$$

So the roots of $f(\rho)$ lie along the curve

$$|(\rho w_\mu)^\nu| = |\theta_a e^{-\rho w_\mu (a_2 - a)}| \tag{11}$$

at intervals whose lengths approach $2\pi/(a_2 - a)$ as ρ becomes infinite. For large values of ρ these roots are all simple, for it is obvious that $f(\rho)$ and $f'(\rho)$ cannot both vanish for the same value of ρ when ρ is large.

In order to prove that in that part of S where $R(\rho w_\mu) \leqq 0$ the roots of Δ_0 are asymptotically equal to those of $f(\rho)$, it is necessary to show that $|f(\rho)|$ remains uniformly away from zero when ρ is uniformly away from a root. For this purpose let T be the region included between the curves defined by the relation

$$|(\rho w_\mu)^\nu| = |\theta_a e^{-\rho w_\mu (a_2 - a) \pm \sigma}|.$$

Let U be the region between T and the line $R(\rho w_\mu) = 0$, and let V be the remainder of S for which $R(\rho w_\mu) < 0$. Choose σ as any convenient positive real number. The following inequalities are immediate:

$$\begin{aligned} &\text{for } U, \quad |(\rho w_\mu)^\nu| \geqq e^{\sigma} |\theta_a e^{-\rho w_\mu (a_2 - a)}|, \\ &\text{for } V, \quad |(\rho w_\mu)^\nu| \leqq e^{-\sigma} |\theta_a e^{-\rho w_\mu (a_2 - a)}|; \end{aligned}$$

from which it follows that

$$\begin{aligned} &\text{for } U, \quad |f(\rho)| \geqq (e^{\sigma} - 1) |\theta_a e^{-\rho w_\mu (a_2 - a)}|, \text{ or } (1 - e^{-\sigma}) |(\rho w_\mu)^\nu|, \\ &\text{for } V, \quad |f(\rho)| \geqq (1 - e^{-\sigma}) |\theta_a e^{-\rho w_\mu (a_2 - a)}|. \end{aligned} \tag{12}$$

For ρ in T it is necessary to proceed differently. Let ρ_0 be any root of $f(\rho) = 0$, and call those points of T which are at least as near to ρ_0 as to any other root of $f(\rho) = 0$, the domain of ρ_0. Now the strip T is asymptotically of constant width, and it is easily shown that this width is $2\sigma/(a_2 - a)$. On the other hand the distance between two roots of $f(\rho) = 0$ was shown to be asymptotically equal to $2\pi/(a_2 - a)$, so that finally if ρ_0 is far enough out the distance of any point of the domain of ρ_0 from ρ_0 will be less than $2(\sigma + \pi)/(a_2 - a)$. In this domain $f(\rho)$ can be written

$$\begin{aligned} f(\rho) &\equiv f(\rho) - f(\rho_0) \equiv (\rho w_\mu)^\nu - (\rho_0 w_\mu)^\nu + \theta_a e^{-\rho w_\mu (a_2 - a)} - \theta_a e^{-\rho_0 w_\mu (a_2 - a)} \\ &\equiv w_\mu^\nu (\rho - \rho_0)(\rho^{\nu-1} + \rho^{\nu-2}\rho_0 + \cdots + \rho_0^{\nu-1}) \\ &\qquad + \theta_a e^{-\rho w_\mu (a_2 - a)} (1 - e^{(\rho - \rho_0) w_\mu (a_2 - a)}). \end{aligned} \tag{13}$$

In T the absolute value of the first term is less than

$$\frac{2(\sigma + \pi)}{a_2 - a} \nu \left| \rho_0 + \frac{2(\sigma + \pi)}{a_2 - a} \right|^{\nu - 1},$$

and for sufficiently large values of ρ this is less than a constant times $|\rho_0|^{\nu-1}$. Moreover, since in T, $|\rho w_\mu|^\nu$ is between two constant multiples of $|e^{-\rho w_\mu(a_2-a)}|$, it follows that the first term is less than a constant times

$$|e^{-[(\nu-1)/\nu]\rho w_\mu(a_2-a)}|.$$

In the parenthesis in the last term let

$$(\rho - \rho_0)\, w_\mu = r(\cos\phi + i \sin\phi), \qquad \text{where} \qquad -\pi \leqq \phi \leqq \pi.$$

The parenthesis becomes

$$(1 - e^{r(a_2-a)(\cos\phi + i\sin\phi)}),$$

which can vanish only at the points given by

$$\cos\phi = 0, \qquad \sin\phi = \frac{2l\pi}{r(a_2 - a)}$$

and the point ρ_0. In the domain of ρ_0 there is only one of these points, namely ρ_0 itself, and for that portion of the domain whose points are at a distance not less than $\delta > 0$ from ρ_0 the absolute value of the parenthesis as a function of r and ϕ is continuous and positive in a closed region. Hence it has a positive minimum which may be denoted by η. The last term of (13) is then greater than a constant times

$$|e^{-\rho w_\mu(a_2-a)}|$$

for that portion of the domain of ρ_0 at a distance from ρ_0 greater than δ. This, combined with the inequality on the first term, gives the result that for large values of ρ in T and at a distance from a root of $f(\rho) = 0$ greater than or equal to $\delta > 0$,

$$|f(\rho)| > |e^{-\rho w_\mu(a_2-a)}| K_0, \tag{14}$$

where K_0 is a positive constant. Similar inequalities have been proved for any ρ in U or V, so that this, with a new constant K_1, may be assumed to hold for all that part of S for which $R(\rho w_\mu) \leqq 0$ for large values of ρ at a distance from a root greater than or equal to δ.

From (10) is obtained the relation

$$\Delta_0 = f(\rho) + \frac{E}{\rho} e^{-\rho w_\mu(a_2-a)} + [\theta_b]\, e^{\rho w_\mu(b-a_2)},$$

which can be factored into

$$\Delta_0 = f(\rho)\left(1 + \frac{E}{\rho}\frac{e^{-\rho w_\mu(a_2-a)}}{f(\rho)} + \frac{[\theta_b]\, e^{\rho w_\mu(b-a_2)}}{f(\rho)}\right). \tag{15}$$

Consider the portions of T, U, and V, bounded by two circular arcs $|\rho| = \rho'$ and $|\rho| = \rho''$, $(\rho' < \rho'')$, such that no point of either arc is at a distance less than δ from a root of $f(\rho) = 0$. Call such a region $\bar{S}$. It will be shown,

by considering the change in arc Δ_0 as ρ traces the boundary of $\bar{S}$, that the number of roots of Δ_0 and of $f(\rho)$ in $\bar{S}$ is the same, provided the innermost boundary of $\bar{S}$ is taken far enough out. Consider first the last term of the parenthesis in (15). Throughout the region $\bar{S}$ the numerator is bounded. In U the denominator, $f(\rho)$, can be replaced by $(1 - e^{-\sigma})(\rho w_\mu)^\nu$ from inequality (12), and so for large values of ρ in U this last term is small. From the second of the inequalities (12) it can similarly be shown that this last term is also small for large values of ρ in V, while from inequality (14) the same result is obtained for the boundary lines of $\bar{S}$ where they cross T, provided the innermost boundary of $\bar{S}$ is far enough out. On the other hand, from inequalities (12) and (14) the expression $e^{-\rho w_\mu(a_2-a)}/f(\rho)$ is bounded on the boundary of $\bar{S}$, and when it is multiplied by E/ρ the result is small when the innermost boundary of $\bar{S}$ is far enough out. Under these conditions, then, the parenthesis in (15) can be written as $1 + \eta$, where η is small enough so that the arc of $1 + \eta$ will return to its original value when ρ traces the boundary of $\bar{S}$. It follows that the change in arc Δ_0 is the same as the change in arc $f(\rho)$, so that Δ_0 and $f(\rho)$ have the same number of roots in $\bar{S}$.

Now exactly the same result is obtained if ρ traces the circumferences of circles of arbitrarily small radius about the roots of $f(\rho)$, and this, combined with the previous result, shows that in that part of the region S for which $R(\rho w_\mu) \leqq 0$, the roots of Δ_0 are asymptotically equal to those of $f(\rho)$.

An important inequality for Δ can be deduced from formula (15). For ρ large and at a distance from a root greater than δ inequality (14) holds. For T and V the exponential can be replaced by a constant times $(\rho w_\mu)^\nu$, while for U the resulting inequality for $f(\rho)$ has already been obtained in (12). The parenthesis in (15) can be made nearly equal to 1, and in particular, greater than $1/2$, by taking ρ large enough; so that the inequality is obtained:

$$|\Delta_0| > |\rho w_\mu|^\nu K_2,$$

in which K_2 is a constant depending only on δ. The result of applying this to Δ is (see (10))

$$|\Delta| > |(\rho w_\mu)^\nu e^{\rho w_\mu(a_2-a)}| K_3, \tag{16}$$

in which K_3 is a constant depending only on δ. This holds for ρ in that part of S where $R(\rho w_\mu)$ is less than or equal to zero and greater than some fixed $\bar{\rho}$, and at a distance from a root of $f(\rho)$ at least as great as δ. Equally important is the result, easily deducible from the same formulas, that under the same conditions Δ is greater than a constant depending only on δ.

$R(\rho w_\mu) \geqq 0$. By entirely similar reasoning it is found that for that part of S where $R(\rho w_\mu) \geqq 0$ the roots of Δ are asymptotically equal to those of

$$(\rho w_\mu)^\nu + \theta_b e^{\rho w_\mu(b-a_2)},$$

which lie along the curve

$$|(\rho w_\mu)^\nu| = |\theta_b\, e^{\rho w_\mu (b-a_2)}|$$

at intervals whose lengths approach $2\pi/(b-a_2)$, and which are all simple for large values of ρ. Inequalities on $f(\rho)$, Δ_0, and Δ, similar to those above are found also in this case.

Finally the odd order case with k points is in general, i. e., when certain constants do not vanish, merely a combination of the cases already considered. The characteristic equation $\Delta = 0$ is as given in (6). The coefficient of $(\rho w_\mu)^\nu$ can be brought to the form (8) by dividing by $e^{\rho w_\mu (a_3-a)}$ and setting $\rho w_\mu (a_3 - a_2) = z$. It is now assumed that none of the constants $\theta_1, \theta_2, \theta_{k-1}, \theta_k$, is zero. It follows then, that the roots of this coefficient are all in a certain region R lying along the line $R(\rho w_\mu) = 0$, and that for ρ at a distance from a root of this coefficient not less than δ the coefficient is in absolute value greater than $|e^{\rho w_\mu (a_2-a)}| K_4$, in which K_4 is a constant dependent only on δ. For convenience let this coefficient be denoted by Δ_1. Then the inequality is

$$|\Delta_1| > |e^{\rho w_\mu (a_2-a)}| K_4, \tag{17}$$

and Δ is

$$\Delta = [\theta_1] + [\theta_k]\, e^{\rho w_\mu (b-a)} + (\rho w_\mu)^\nu \Delta_1.$$

This can be written

$$\Delta = (\rho w_\mu)^\nu \Delta_1 \left(1 + \frac{[\theta_1] + [\theta_k]\, e^{\rho w_\mu (b-a)}}{(\rho w_\mu)^\nu \Delta_1}\right). \tag{18}$$

Now in R the exponentials are bounded. Hence, if in R any region be drawn, such that no part of the boundary is at a distance from a root of Δ_1 less than $\delta > 0$, then on the boundary Δ_1 remains uniformly away from zero, and if the region be far enough out the parenthesis in (18) has, on the boundary, the form $1 + \eta$, where $|\eta| < 1$. From this can be deduced the fact that the roots of Δ in R are asymptotically equal to those of Δ_1.

Outside of R for that part of S in which $R(\rho w_\mu) < 0$ the roots of Δ are asymptotically equal to those of

$$f(\rho) \equiv (\rho w_\mu)^\nu + \frac{\theta_1}{\theta_2} e^{-\rho w_\mu (a_2-a)} = 0,$$

and for that part of S in which $R(\rho w_\mu) > 0$ they are asymptotically equal to those of

$$(\rho w_\mu)^\nu + \frac{\theta_k}{\theta_{k-1}} e^{\rho w_\mu (b-a_{k-1})} = 0.$$

To prove the first of these statements, write

$$\frac{\Delta}{\Delta_1} = f(\rho) + \frac{[\theta_1]}{\Delta_1} - \frac{\theta_1}{\theta_2} e^{-\rho w_\mu (a_2-a)} + \frac{[\theta_k]\, e^{\rho w_\mu (b-a)}}{\Delta_1}$$

$$(19) \qquad = f(\rho) + \frac{[0] - \theta_1 \sum_{j=3}^{k-1} [\theta_j]\, e^{\rho w_\mu (a_j - a_2)}}{\Delta_1 \theta_2} + \frac{[\theta_k]\, e^{\rho w_\mu (b-a)}}{\Delta_1}$$

$$= f(\rho)\left(1 + \frac{[0]}{\Delta_1 \theta_2 f(\rho)} - \frac{\theta_1 \sum_{j=3}^{k-1} [\theta_j]\, e^{\rho w_\mu (a_j - a_2)}}{\Delta_1 \theta_2 f(\rho)} + \frac{[\theta_k]\, e^{\rho w_\mu (b-a)}}{\Delta_1 f(\rho)}\right).$$

The roots of $f(\rho)$ have already been found, for it is the $f(\rho)$ of (11). To prove the roots of Δ asymptotically equal to those of $f(\rho)$, it is obviously sufficient to prove that the parenthesis above has the form $1 + \eta$, for ρ on the boundary of any region outside of R and in that part of S where $R(\rho w_\mu) < 0$, and such that the distance of any point of the boundary from a root of $f(\rho)$ is not less than $\delta > 0$, provided that this region is taken far enough out from the origin. For such values of ρ the inequalities (12), (14), and (17) hold, and they will be used to show that the parenthesis has that form. From (14) and (17) is obtained $|f(\rho)\Delta_1| > K_1 K_4$, and since the symbol $[0]$ is the same thing as E/ρ it is evident that the term $[0]/\Delta_1 \theta_2 f(\rho)$ is small for ρ large. In the next term the same inequality on $f(\rho)\Delta_1$ is used for T and V, since in these regions, for ρ large, $R(\rho w_\mu)$ is large and negative, and the numerator, and so the whole term, is small. For U the inequality $|f(\rho)| \geqq (1 - e^{-\sigma})|(\rho w_\mu)^\nu|$ from (12) is used for $f(\rho)$. The remainder of the term may be written in the form

$$E e^{-\rho w_\mu \{(a_2 - a) - (a_3 - a_2)\}} = E e^{-\rho w_\mu (a_2 - a)[1 - (a_3 - a_2)/(a_2 - a)]}.$$

Now in U the inequality

$$|(\rho w_\mu)^\nu| \geqq e^\sigma |\theta_a\, e^{-\rho w_\mu (a_2 - a)}|$$

holds, from which it follows that

$$|e^{-\rho w_\mu (a_2 - a)[1 - (a_3 - a_2)/(a_2 - a)]}| \leqq |(\rho w_\mu)^{\nu[1 - (a_3 - a_2)/(a_2 - a)]}|\, \overline{K},$$

where $\overline{K}$ is an absolute constant. The exponent of this power of ρ is certainly less than ν, so that because of the factor $|(\rho w_\mu)^\nu|$ from $f(\rho)$ in the denominator, the term approaches zero as ρ becomes infinite. By reason of inequality (17) the last term of the parenthesis can obviously be written $E/f(\rho)$. In T and V inequality (16) shows that this is small for large values of ρ, while the inequality $|f(\rho)| \geqq (1 - e^{-\sigma})|(\rho w_\mu)^\nu|$ from (12) serves the same purpose for U. Thus the parenthesis has been shown to be of the required form for the values of ρ in question, and so the roots of Δ are asymptotically equal to those of $f(\rho)$ for that part of S outside of R for which $R(\rho w_\mu) < 0$.

The proof for the remaining part of S on the other side of R is similar.

It is to be noted that the above reasoning applies to the whole of that part of S in which $R(\rho w_\mu) \leqq 0$, even including the strip R, provided that ρ is at a distance greater than δ from a root of $f(\rho)$ or Δ_1, and is large enough. Hence from (19) can be obtained the following inequality which holds for all ρ's in absolute value greater than a certain $\bar{\rho}$ and at a distance from a root of Δ greater than δ:

$$|\Delta| > |\Delta_1| \cdot |f(\rho)| K_5, \tag{20}$$

in which K_5 is a constant depending only on δ.

In particular this shows, by reason of inequalities (14) and (17), that if in this region ρ remains uniformly away from a root, then Δ remains uniformly away from zero.

Another inequality for Δ can easily be deduced from the above. First replace $|\Delta_1|$ by $|e^{\rho w_\mu(a_2-a)}| K_4$ from inequality (17). Then for the region U replace $|f(\rho)|$ by $(1 - e^{-\sigma})|(\rho w_\mu)|^\nu$ from (12), and for the regions T and V replace it by $|e^{-\rho w_\mu(a_2-a)}| K_1$ from (14). But in the regions T and V the exponential can be replaced in this inequality by a constant times $|(\rho w_\mu)|^\nu$, so that finally the following is obtained:

$$|\Delta| > |(\rho w_\mu)^\nu e^{\rho w_\mu(a_2-a)}| K_6, \tag{21}$$

in which K_6 is a constant depending only on δ, and the inequality holds for all values of ρ greater in absolute value than a certain $\bar{\rho}$ and at a distance from a root of Δ greater than δ.

Even order case. When n is even the substitution of the asymptotic expressions for the W's in the determinant $D(\lambda)$ gives

$$D(\lambda) \equiv \prod_{i=1}^{n} \rho^{h_i} \prod_{j=\mu+2}^{n} e^{\rho w_j(b-a)} \Delta,$$

in which Δ is that determinant which has for the element in the ith row and the jth column, $[\alpha_{1i}]\, w_j^{h_i}$ for $j = 1, 2, \cdots, \mu - 1$, and $[\alpha_{ki}]\, w_j^{h_i}$ for $j = \mu + 2, \mu + 3, \cdots, n$, and

$$\begin{aligned} D_i &\equiv w_\mu^{h_i}([\alpha_{1i}] + [\alpha_{ki}]\, e^{\rho w_\mu(b-a)} + \sum_{l=2}^{k-1} (\rho w_\mu)^{h_{li}-h_i}[\alpha_{li}]\, e^{\rho w_\mu(a_l-a)}) \quad (j = \mu), \\ H_i &\equiv (-w_\mu)^{h_i}([\alpha_{1i}] + [\alpha_{ki}] e^{-\rho w_\mu(b-a)} + \sum_{l=2}^{k-1} (-\rho w_\mu)^{h_{li}-h_i}[\alpha_{li}]\, e^{-\rho w_\mu(a_l-a)}) \\ &\qquad\qquad (j = \mu + 1) \end{aligned} \tag{21a}$$

where $n = 2\mu$.

It is evident from the definition of the bracket symbol that in general this determinant expression for Δ can be expanded in the form

$$\Delta \equiv P_1(\rho w_\mu)^{2\nu} + P_2(\rho w_\mu)^\nu + P_3, \tag{22}$$

where the highest and lowest order terms of the P's are

$$
\begin{aligned}
&P_1\colon [\theta_{11}]\, e^{\rho w_\mu (a_{k-1}-a_2)} \quad \text{and} \quad [\theta_{12}]\, e^{-\rho w_\mu (a_{k-1}-a_2)};\\
&P_2\colon [\theta_{21}]\, e^{\rho w_\mu (b-a_2)} \quad \text{and} \quad [\theta_{22}]\, e^{-\rho w_\mu (b-a_2)}, \quad \text{if} \quad b-a_2 > a_{k-1}-a,\\
&\quad \text{or} \quad [\theta_{31}]\, e^{\rho w_\mu (a_{k-1}-a)} \quad \text{and} \quad [\theta_{32}]\, e^{-\rho w_\mu (a_{k-1}-a)}, \quad \text{if} \quad b-a_2 < a_{k-1}-a;\\
&P_3\colon [\theta_{41}]\, e^{\rho w_\mu (b-a)} \quad \text{and} \quad [\theta_{42}]\, e^{-\rho w_\mu (b-a)}.
\end{aligned} \tag{23}
$$

The number ν is the exponent of the highest power of ρ appearing in a D_i or H_i.

We shall now prove the

THEOREM. *If none of the θ's in* (23) *are zero, then $D(\lambda)$ has an infinite number of roots, and there exist two absolute constants ω and Ω such that the number of roots of $D(\lambda)$ in any circle about the origin of radius r is less than Ωr and greater than ωr, and finally for values of λ uniformly away from its roots, $D(\lambda)$ remains uniformly away from zero.*

This last statement is given more explicitly by inequality (25).

Should the number ν be negative, Δ reduces formally to Birkhoff's even order case,* and the problem of distribution of roots will not be considered farther here. If ν is zero, Δ becomes merely a sum of exponentials which when multiplied through by the non-vanishing factor $e^{\rho w_\mu (b-a)}$ all have real parts of the same sign, and so are reduced to the form (8) already treated. In either of these cases the formal expansion (22) is not needed. But when ν is positive we make use of this expansion, which tacitly assumes that at least two of the D_i's and H_i's actually involve this power of ρ. This is, of course, equivalent to assuming that the highest order derivative in the auxiliary conditions at points other than the end points appears in at least two conditions. We also assume that none of the θ's in (23) are zero, which is to say that certain of the determinants from the matrix of coefficients of the auxiliary conditions must not vanish. The method here used can be applied equally well with very slight modifications to many of the cases thus excluded, but a detailed study of the more specialized cases is beyond the scope of this paper.

The more general even order case, then, is treated as follows: The function P_1 can be reduced to the type (8) by multiplying through by $e^{\rho w_\mu (a_{k-1}-a_2)}$ and making the proper substitution. Its roots, then, all lie in a region R, and the proof that the roots of Δ in R are asymptotically equal to those of P_1 is immediate. To find the roots outside of R write

$$\frac{\Delta}{P_1} = (\rho w_\mu)^{2\nu} + (\rho w_\mu)^{\nu}\frac{P_2}{P_1} + \frac{P_3}{P_1}.$$

The line $R(\rho w_\mu) = 0$ is now not the bisecting ray of each of the S regions

* Birkhoff, loc. cit., p. 387.

as in the odd order case, but the line of demarcation between two of them. Hence the region R overlaps the two S regions into which the λ-plane is carried by the transformation $\lambda = \rho^n$. Consider first the region S_1 in which $R(\rho w_\mu) < 0$. The above expression can be written in the form

$$\frac{\Delta}{P_1} = \left\{ (\rho w_\mu)^\nu + \frac{[\theta_{32}]}{[\theta_{12}]} e^{-\rho w_\mu (a_2 - a)} \right\} \left\{ (\rho w_\mu)^\nu + \frac{[\theta_{42}]}{[\theta_{32}]} e^{-\rho w_\mu (b - a_{k-1})} \right\} + (\rho w_\mu)^\nu Q_1 + Q_2.$$

Suppose that $a_2 - a$ is not equal to $b - a_{k-1}$ and, for convenience, let $a_2 - a > b - a_{k-1}$. Then the term of highest order in Q_1 is

$$[\theta]\, e^{-\rho w_\mu (b - a_{k-1})},$$

and in Q_2,

$$[\theta]\, e^{-\rho w_\mu (a_2 - a)}.$$

The expressions Q_1 and Q_2 are merely those terms which must be added to the product of the two factors in order to make the result equal to Δ/P_1.

The roots of the two parentheses are found by the methods already developed, and it can be shown by means of inequalities of the type (12) and (14) that the roots of Δ in S_1 and outside of R are asymptotically equal to those of the two parentheses.

Similarly the roots of Δ in S_2 and outside of R are asymptotically equal to those of two parentheses of the same type, but with positive instead of negative exponents.

The following inequality is easily found for large values of ρ which are in S_1 and at a distance from a root of Δ greater than δ:

$$|\Delta| > \left| (\rho w_\mu)^\nu + \frac{[\theta_{32}]}{[\theta_{21}]} e^{-\rho w_\mu (a_2 - a)} \right| \cdot \left| (\rho w_\mu)^\nu + \frac{[\theta_{42}]}{[\theta_{32}]} e^{-\rho w_\mu (b - a_{k-1})} \right| \cdot |P_1| K_7,$$

in which K_7 is a constant depending only on δ. If

$$\bar{\Delta} = \Delta e^{\rho w_\mu (b - a)},$$

the inequality for $\bar{\Delta}$ is

$$|\bar{\Delta}| > |P_1\, e^{\rho w_\mu (a_{k-1} - a_2)}| \cdot |(\rho w_\mu)^\nu\, e^{\rho w_\mu (a_2 - a)} + \frac{[\theta_{32}]}{[\theta_{12}]}| \cdot |(\rho w_\mu)^\nu\, e^{\rho w_\mu (b - a_{k-1})} + \frac{[\theta_{42}]}{[\theta_{32}]}| K_7. \tag{24}$$

It can also be proved that for large values of ρ in S_1 which are at a distance greater than δ from a root of $\bar{\Delta}$, the absolute value of $\bar{\Delta}$ remains greater than a positive constant depending only on δ.

When $a_2 - a = b - a_{k-1}$ the two systems of roots lie along two curves of the type (11) in which both have the same exponential, although in general

the coefficients are different. If these coefficients should have the same absolute value the roots all lie along one curve, and if they are identical the roots are all double. An inequality like (24) holds in this case also, the only difference being in the θ's and the value of K_7. From (24) it is easy to deduce the inequalities

$$
\begin{aligned}
|\bar{\Delta}| &> |(\rho w_\mu)^{2\nu} e^{\rho w_\mu (a_2 - a)} e^{\rho w_\mu (b - a_{k-1})}| K_8, \\
&> |(\rho w_\mu)^{\nu} e^{\rho w_\mu (a_2 - a)}| K_9, \\
&> |(\rho w_\mu)^{\nu} e^{\rho w_\mu (b - a_{k-1})}| K_{10},
\end{aligned} \tag{25}
$$

which hold under the same conditions as (24), and in which K_8, K_9, and K_{10} depend only on δ.

4. The expansion problem

The expansion theorem is as follows:

THEOREM. *Given $f(x)$, any real function which together with its first ν derivatives (or if $\nu < 1$ with its first derivative) is continuous in the closed interval (a, b), then its expansion in terms of the residues of the Green's function (3) converges to $f(x)$ at every interior point of (a, b), if the auxiliary conditions have the following properties:*

In the odd order case

Neither θ_1 nor θ_k in (6) is zero.

If $\nu \geqq 1$ then in the coefficient of $(\rho w_\mu)^\nu$ neither θ_2 nor θ_{k-1} is zero.

In the even order case

None of the θ's in (23) are zero.

The intervals $b - a_2$ and $a_{k-1} - a$ are longer than any other of the subintervals of (a, b) included between any two of the points $a, a_2, a_3, \cdots, a_{k-1}, b$.

The essential purpose of this last condition is merely that the second point from either end shall be farther away from that end than the first point from the other end is from that end. The conditions on the θ's require merely that certain determinants from the matrix of the constants in the auxiliary conditions do not vanish. The condition on the θ's in the even order case is stated in the above form in order to have it as simple as possible. If $\nu < 1$ it is not necessary that all these θ's be different from zero.

To prove the theorem it is necessary to consider separately the odd and even order cases. But before that is done it is convenient to introduce a new form of the expression for $\bar{G}(x, s; \lambda)$ given in (2). If $\bar{G}$ be expanded in terms of the first row of the numerator determinant it can be written

$$
\bar{G}(x, s; \lambda) \equiv \pm \frac{1}{2} \sum_{i=1}^{n} y_i(x) \bar{y}_i(s),
$$

in which

$$
\bar{y}_i(s) \equiv \frac{e^{-\rho w_i (s - a)} w_i}{n \rho^{n-1}} \left(1 + \frac{\psi_1(s)}{\rho w_i} + \frac{\psi_2(s)}{(\rho w_i)^2} + \cdots + \frac{\psi_{n-1}(s)}{(\rho w_i)^{n-1}} + \frac{E}{\rho^n} \right). \tag{26}
$$

This expression for the $\bar{y}$'s in so far as it differs from the one given by Birkhoff* is due to Milne.† In the work to follow, if $\nu < 1$, Birkhoff's form would be sufficient, and the convergence proof is no more complicated than in the two point case. The complications necessary in case $\nu \geqq 1$, are due to the necessity of using the more explicit formulas of Milne.

The above form of $\bar{G}$ gives for $W_i(\bar{G})$

$$W_i(\bar{G}) \equiv -\frac{1}{2}\sum_{j=1}^{n} W_{1i}(y_j)\bar{y}_j(s) \pm \frac{1}{2}\sum_{j=1}^{n} W_{2i}(y_j)\bar{y}_j(s) \pm \cdots$$
$$\pm \frac{1}{2}\sum_{j=1}^{n} W_{k-1i}(y_j)\bar{y}_j(s) + \frac{1}{2}\sum_{j=1}^{n} W_{ki}(y_j)\bar{y}_j(s),$$

in which the plus sign is taken for values of s to the left of the point at which the term in question applies, and the negative sign for values of s to the right.

Odd order case. $R(\rho w_\mu) \leqq 0$. The convergence of the integral (5) is now to be studied. It has already been separated into the sum of two integrals, one in S_1 and one in S_2. In the odd order case it is necessary to separate each still farther into the sum of two integrals, one in that part of S where $R(\rho w_\mu) < 0$, and the other in that part where $R(\rho w_\mu) > 0$. The successive paths of integration are arcs of circles $|\rho| =$ constant extending across both the regions S_1 and S_2, and so chosen that no point of such an arc is at a distance less than a preassigned $\delta > 0$ from a root of D. That such arcs can be drawn in such a way that the sum of the orders of the roots between two successive arcs is not greater than $2k$ when δ is chosen small enough, follows from the distribution of the roots of Δ as determined earlier in this paper. The parts of these arcs in S_1 and S_2 will be denoted by γ_1 and γ_2 as in (5). Let S represent either of these regions. Then let B be the point where the line $R(\rho w_\mu) = 0$ in S cuts one of these arcs, and let A be the other end of the arc that crosses the region where $R(\rho w_\mu) < 0$, and C the other end of the arc that crosses the region where $R(\rho w_\mu) > 0$. The convergence of the integral over the arc AB will be studied first; that is, the integral

$$I_{AB} \equiv \frac{1}{2\pi i}\int_{AB}\int_a^b n\rho^{n-1} G(x, s; \rho^n) f(s)\, ds\, d\rho. \tag{27}$$

Before making use of the expression (3) for $G(x, s; \rho^n)$ it is convenient to alter the last column by adding to it each of the columns 1 to μ multiplied by the term $\frac{1}{2}\bar{y}_i(s)$ with corresponding index, and each of the columns $\mu + 1$ to n multiplied by the term $-\frac{1}{2}\bar{y}_i(s)$ with corresponding index. The resulting last column is:

1st element, $\sum_{i=1}^{\mu} y_i(x)\bar{y}_i(s)$ for $y < x$, or $-\sum_{i=\mu+1}^{n} y_i(x)\bar{y}_i(s)$ for $y > x$;

* Birkhoff, loc. cit., p. 391.

† Milne, loc. cit., p. 167.

$(1+i)$th element,

$$(28)\quad -\sum_{j=\mu+1}^{n} W_{1i}(y_j)\bar{y}_j(s) + \left\{\begin{array}{l} -\sum_{j=\mu+1}^{n} W_{2i}(y_j)\bar{y}_j(s) \\ \sum_{j=1}^{\mu} W_{2i}(y_j)\bar{y}_j(s) \end{array}\right\} + \cdots$$
$$+ \left\{\begin{array}{l} -\sum_{j=\mu+1}^{n} W_{k-1,i}(y_j)\bar{y}_j(s) \\ \sum_{j=1}^{\mu} W_{k-1,i}(y_j)\bar{y}_j(s) \end{array}\right\} + \sum_{j=1}^{\mu} W_{ki}(y_j)\bar{y}_j(s),$$

in which the upper term of each brace is taken for values of s to the right of the point to which that term applies, and the lower for values of s to the left.

The asymptotic solutions are now substituted in the expression G, and then the factor ρ^{h_i} of D is divided into the $(1+i)$th row of the numerator, and the factor $e^{\rho w_j(b-a)}$ into the jth column. The factor -1 is divided into the last column. Thus is obtained

$$G(x, s; \rho^n) = \frac{\bar{N}}{n\rho^{n-1}\Delta},$$

in which $\bar{N}$ is the determinant Δ bounded on the top by $e^{\rho w_i(x-a)}[1]$ with corresponding index for the columns 1 to μ, and by $e^{\rho w_i(x-b)}[1]$ with corresponding index for the columns $\mu+1$ to n. On the right the ith row is bounded by F_i and the upper right-hand corner is filled in with F, where F_i is

$$(29)\quad -\sum_{j=\mu+1}^{n}[\alpha_{1i}\, w_j^{h_i+1}]e^{\rho w_j(a-s)} + \sum_{l=2}^{k-1}\rho^{h_{li}-h_i}\left\{\begin{array}{l} -\sum_{j=\mu+1}^{n}[\alpha_{li}\, w_j^{h_{li}+1}]\, e^{\rho w_j(a_l-s)} \\ +\sum_{j=1}^{\mu}\alpha_{li}\, w_j^{h_{li}+1}]\, e^{\rho w_j(a_l-s)} \end{array}\right\}$$
$$+\sum_{j=1}^{\mu}[\alpha_{ki}\, w_j^{h_i+1}]\, e^{\rho w_j(b-s)},$$

and F is

$$\sum_{j=1}^{\mu} e^{\rho w_j(x-s)}[w_j], \text{ for } s<x, \text{ and } -\sum_{j=\mu+1}^{n} e^{\rho w_j(x-s)}[w_j], \text{ for } s>x;$$

and for convenience we give B_i, the ith element in the μth column of Δ: B_i is

$$w_\mu^{h_i}\left([\alpha_{1i}] + \sum_{l=2}^{k-1}(\rho w_\mu)^{h_{li}-h_i}[\alpha_{li}]\, e^{\rho w_\mu(a_l-a)} + [\alpha_{ki}]\, e^{\rho w_\mu(b-a)}\right).$$

This expression for $G(x, s; \rho^n)$ can now be substituted into (27), and since s occurs only in the last column of $\bar{N}$, the function $f(s)$ can be multiplied into this column, and the s integration performed. It is then to be noted that all terms of the determinant save possibly those in the μth and last

columns are bounded. Moreover the cofactor of F is $-\Delta$, and so the determinant can be expanded to give

$$
\begin{aligned}
I_{AB} \equiv \frac{1}{2\pi i}\int_{AB}\int_a^b - Ff(s)\,ds & \qquad (a)\\
+\frac{1}{2\pi i}\int_{AB}\left(\sum_{j=1}^{\mu-1}\sum_{r=1}^{n}\sum_{l=1}^{n} E_{jrl}\, e^{\rho w_j(x-a)}\frac{B_r}{\Delta}\int_a^b F_l f(s)\,ds\right)d\rho & \qquad (b)\\
+\frac{1}{2\pi i}\int_{AB}\left(\sum_{j=\mu+1}^{n}\sum_{r=1}^{n}\sum_{l=1}^{n} E_{jrl}\, e^{\rho w_j(x-b)}\frac{B_r}{\Delta}\int_a^b F_l f(s)\,ds\right)d\rho & \qquad (c)\\
+\frac{1}{2\pi i}\int_{AB}\left(\sum_{r=1}^{n}\frac{E_r}{\Delta}\, e^{\rho w_\mu(x-a)}\int_a^b F_r f(s)\,ds\right)d\rho. & \qquad (d)
\end{aligned}
\tag{30}
$$

It will be remembered that E is a generic notation for functions of ρ (and other variables) uniformly bounded for large values of ρ. In particular, in (b) and (c), E_{jrl} is identically zero for $r = 1$.

The integral (a) will be treated first. It is necessary to separate it into the sum of two integrals, one from a to x and the other from x to b. The result of integrating these by parts with respect to s gives $(\mu/4n)f(x) + \epsilon$ and $((\mu-1)/4n)f(x) + \epsilon$ respectively, for, as is easily seen,

$$
\begin{aligned}
&\int_{AB}\rho^\alpha\, e^{\rho w_j(x-a)}\,d\rho, \qquad \text{for} \qquad j < \mu,\\
&\text{and}\\
&\int_{AB}\rho^\alpha\, e^{\rho w_j(x-b)}\,d\rho, \qquad \text{for} \qquad j > \mu,
\end{aligned}
\tag{31}
$$

approach zero as the radius of the arc AB becomes infinite, for α any real constant, and x distinct from a and b. One also easily shows that

$$
\int_{AB}\frac{e^{\rho w_\mu(x-a)}}{\rho}\,d\rho \tag{32}
$$

approaches zero as this radius becomes infinite.

To evaluate (b) and (c) it is only necessary to notice that $1/\Delta$ is bounded, while both D_r and the s integral are certainly less than a constant times ρ^n. Hence (b) and (c) reduce merely to bounded functions multiplied into integrals of the type (31), and so approach zero as the radius of AB becomes infinite.

The evaluation of (d) is more complicated. We now make use of the more explicit asymptotic solutions of Milne, and his expressions for the $\bar{y}$'s, (26). In the formula for F_i in (29), the square bracket $[\alpha_{li}\, w_j^{h\mu+1}]$, for example, has the more explicit form

$$
[\alpha_{li}\, w_j^{h\mu+1}] = w_j^{h\mu+1}\left(\alpha_{li} + \sum_{r=1}^{n-1}\frac{\alpha_{lir}(s)}{(\rho w_j)^r} + \frac{E}{\rho^n}\right).
$$

Consider now

$$\int_a^b F_i f(s)\,ds.$$

It can be broken up into the sum of several integrals, since F_i is a sum. Of these integrals consider first the one with the exponential involving a_l ($l \neq 1, k$). When the more explicit form of the bracket symbol given above is used this becomes

$$\begin{aligned}
&\rho^{h_{li}-h_i}\Bigg(\int_a^{a_l}\sum_{j=1}^{\mu}\alpha_{li}\,w_j^{h_{li}+1}\,e^{\rho w_j(a_l-s)}f(s)\,ds \\
&\qquad\qquad +\int_{a_l}^{b}-\sum_{j=\mu+1}^{n}\alpha_{li}\,w_j^{h_{li}+1}\,e^{\rho w_j(a_l-s)}f(s)\,ds \qquad (a)\\
&\quad+\int_a^{a_l}\sum_{j=1}^{\mu}\sum_{r=1}^{n-1}\frac{\alpha_{lir}(s)}{(\rho w_j)^r}\,e^{\rho w_j(a_l-s)}\,w_j^{h_{li}+1}f(s)\,ds \\
&\qquad\qquad +\int_{a_l}^{b}-\sum_{j=\mu+1}^{n}\sum_{r=1}^{n-1}\frac{\alpha_{lir}(s)}{(\rho w_j)^r}\,e^{\rho w_j(a_l-s)}\,w_j^{h_{li}+1}f(s)\,ds \qquad (b)\\
&\quad+\int_a^{a_l}\sum_{j=1}^{\mu}\frac{E}{\rho^n}\,e^{\rho w_j(a_l-s)}\,w_j^{h_{li}+1}f(s)\,ds \\
&\qquad\qquad +\int_{a_l}^{b}-\sum_{j=\mu+1}^{n}\frac{E}{\rho^n}\,e^{\rho w_j(a_l-s)}\,w_j^{h_{li}+1}f(s)\,ds\Bigg). \qquad (c)
\end{aligned} \tag{33}$$

Now integrate (a) by parts and add the results, obtaining

$$\begin{aligned}
\frac{\alpha_{li}}{\rho}\Bigg(&f(a_l)\sum_{j=1}^{n}(-w_j^{h_{li}}) + \sum_{j=1}^{\mu}w_j^{h_{li}}f(a)\,e^{\rho w_j(a_l-a)} + \sum_{j=\mu+1}^{n}w_j^{h_{li}}f(b)\,e^{\rho w_j(a_l-b)} \\
&+\int_a^{a_l}\sum_{j=1}^{\mu}w_j^{h_{li}}e^{\rho w_j(a_l-s)}f'(s)\,ds + \int_{a_l}^{b}-\sum_{j=\mu+1}^{n}w_j^{h_{li}}\,e^{\rho w_j(a_l-s)}\,f'(s)\,ds\Bigg).
\end{aligned}$$

Since $n > h_{li} > 0$, the coefficient of $f(a_l)$ is zero. The remaining integrals can again be integrated by parts,* and again the terms involving a_l will drop out. This is repeated $h_{li} - h_i$ times, which will give the factor $\alpha_{li}/\rho^{h_{li}-h_i}$ outside, and this is just enough to cancel the factor in (33). Then absolute values are taken, $f^{(h_{li}-h_i-1)}(s)$ is replaced by the upper limit to its absolute value, and the exponentials are again integrated, giving E/ρ. The total result may be written

$$\sum_{j=1}^{\mu}E_{ij}\,\rho^{h_{li}-h_i-1}\,e^{\rho w_j(a_l-a)} + \sum_{j=\mu+1}^{n}E_{ij}\,\rho^{h_{li}-h_i-1}\,e^{\rho w_j(a_l-b)} + \frac{E_i}{\rho}. \tag{34}$$

Next consider the rth term of (b). The factor $1/\rho^r$ can be removed from

* It is to be noted that the number of derivatives here required has been provided for in the hypothesis of the expansion theorem, and also that if $\nu < 1$ all this work with Milne's formulas is unnecessary.

under the sign of integration, and $\alpha_{lir}(s)f(s)$ can be considered as one function. The integral can then be treated as was (a), but it requires only $h_{li} - h_i - r$ integrations by parts and one further integration. The result can then be written in the same form as (34), though in this case, to be sure, the E's are not only bounded but actually approach zero. The terms (c) are obviously of the form E/ρ, so that the whole of (33) can be written in the form (34). Finally a single integration by parts reduces the terms involving the points a and b to the form E/ρ, so that the integral

$$\int_a^b F_i f(s)\, ds$$

can be written as

$$(35) \qquad \sum_{l=2}^{k-1}\left(\sum_{j=1}^{\mu} E_{ijl}\,\rho^{h_{li}-h_i-1}\,e^{\rho w_j(a_l-a)} + \sum_{j=\mu+1}^{n} E_{ijl}\,\rho^{h_{li}-h_i-1}\,e^{\rho w_j(a_l-b)}\right) + \frac{E_i}{\rho}.$$

When this is substituted in (30d) the last term here gives a result of the form (32). The other terms for $j \neq \mu$ give results of the type (31) with the value a_l for x, and since for these terms a_l is distinct from a and b, the result of the ρ integration approaches zero as the radius of AB becomes infinite. The only terms left from (35) then are

$$\sum_{l=2}^{k-1} E_l\,\rho^{h_{li}-h_i-1}\,e^{\rho w_\mu(a_l-a)},$$

which are multiplied by $(E/\Delta)\,e^{\rho w_\mu(x-a)}$ for the ρ integration. If it can be proved that

$$\sum_{l=2}^{k-1} \frac{E_l\,\rho^{h_{li}-h_i-1}\,e^{\rho w_\mu(a_l-a)}}{\Delta}$$

is of the form E/ρ, then this last term will give a result of the form (32), and (30d) will have been shown to approach zero as the radius of AB becomes infinite. Now inequalities (16) and (21) prove this for $\nu > 0$, while for $\nu \leqq 0$ the power of ρ is negative while the exponential is bounded and Δ remains uniformly away from zero, and the same result is obtained.

To sum up the results, for the radius of AB large

$$I_{AB} = \tfrac{1}{4}f(x) + \epsilon.$$

The work for the case $R(\rho w_\mu) > 0$ runs along parallel to this in every detail, and the result is obtained that for the radius of BC large

$$I_{BC} = \tfrac{1}{4}f(x) + \epsilon.$$

This result holds for both S_1 and S_2, so that (1) converges to $f(x)$ in the odd order case.

Even order case. In the even order case the paths of integration are the

same as in the odd order case, but now there is no need of dividing farther the arcs γ_1 and γ_2, since throughout each region S the sign of the real part of ρw_μ is unaltered. The region S_1 in which $R(\rho w_\mu) \leqq 0$ will be considered first, that is the integral

$$I_{\gamma_1} = \frac{1}{2\pi i}\int_{\gamma_1}\int_a^b n\rho^{n-1} G(x, s; \rho^n) f(s)\, ds d\rho .$$

In making use of the expression (3) for $G(x, s; \rho^n)$ we now alter the last column by adding to it each of the columns 1 to μ multiplied by $\frac{1}{2}\bar{y}_i(s)$ with corresponding index, and each of the columns $\mu+1$ to n multiplied by $-\frac{1}{2}\bar{y}_i(s)$ with corresponding index. The asymptotic solutions are then substituted, and the factor ρ^{h_i} of D is divided into the $(1+i)$th row, and the factor $e^{\rho w_j(b-a)}$ into the jth column. The factor $e^{\rho w_{\mu+1}(b-a)}$ is also divided out of the $(\mu+1)$th column and out of D, thus leaving in the denominator

$$\bar{\Delta} \equiv e^{\rho w_\mu (b-a)}\Delta ,$$

since $w_{\mu+1} = -w_\mu$. Thus $G(x, s; \rho^n)$ is found to be equal to $1/(n\rho^{n-1}\bar{\Delta})$ multiplied into a determinant of order $n+1$ in which the element in the first row and jth column is $e^{\rho w_j(x-a)}[1]$ for $j \leqq \mu$ and $e^{\rho w_j(x-b)}[1]$ for $\mu+1 \leqq j \leqq n$. The last element is F, given in (29). In the $(1+i)$th row and jth column the element is $[\alpha_{1i} w_j^{h_i}]$ for $j < \mu$, $[\alpha_{ki} w_j^{h_i}]$ for $\mu+1 < j \leqq n$, while for $j = \mu$, $\mu+1$, and $n+1$ the elements are respectively D_i, $\bar{H}_i$, and F_i, where D_i is as given in (21a), while $\bar{H}_i$ is the H_i given there, multiplied by $e^{\rho w_\mu(b-a)}$, and F and F_i are as given in (29).

The variable s appears only in the last column of this determinant, whence $f(s)$ can be multiplied into this last column and the s integration performed. Since all terms, save possibly the D's, $\bar{H}$'s, and F's, are bounded, the expansion of the determinant takes the form

$$I_{\gamma_1} \equiv \frac{1}{2\pi i}\int_{\gamma_1}\int_a^b - Ff(s)\, ds d\rho \qquad (a)$$

$$+\frac{1}{2\pi i}\int_{\gamma_1}\sum_{i,j,r=1}^{n}\begin{vmatrix} D_i & \bar{H}_i & \int F_i f(s)\, ds \\ D_j & \bar{H}_j & \int F_j f(s)\, ds \\ D_r & \bar{H}_r & \int F_r f(s)\, ds \end{vmatrix}\left(\sum_{l=1}^{\mu-1} E_l\, e^{\rho w_l(x-a)}\right.$$

$$\left. + \sum_{l=\mu+1}^{n} E_l\, e^{\rho w_l(x-b)}\right) d\rho \qquad (b)$$

$$+\frac{1}{2\pi i}\int_{\gamma_1}\sum_{i,j=1}^{n}\begin{vmatrix} e^{\rho w_\mu(x-a)}[1] & e^{\rho w_{\mu+1}(x-b)}[1] & 0 \\ D_i & \bar{H}_i & \int F_i f(s)\, ds \\ D_j & \bar{H}_j & \int F_j f(s)\, ds \end{vmatrix}\frac{E_{ij}}{\Delta}\, d\rho . \qquad (c)$$

By comparing (a) with the similar integral in the odd order case it is obvious that it approaches the limit $f(x)/2$ as the radius of γ_1 becomes infinite. In (b) the determinants can be written in the form $E\rho^{\alpha}$, where $\alpha < 3n$. Since in the parenthesis the exponents have negative real parts on γ_1, (b) falls under the type (31), and so approaches zero as the radius of γ_1 becomes infinite. Finally (c) is expanded in terms of the top row of the determinant, and the result is treated in a manner similar to that used in the odd order case. The expanded form is

$$(36)\quad E_1 e^{\rho w_\mu (x-a)} \sum_{i,j} \frac{1}{\Delta} \begin{vmatrix} \bar{H}_i & \int F_i f ds \\ \bar{H}_j & \int F_j f ds \end{vmatrix} + E_2 e^{\rho w_{\mu+1}(x-b)} \sum_{i,j} \frac{1}{\Delta} \begin{vmatrix} D_i & \int F_i f ds \\ D_j & \int F_j f ds \end{vmatrix}.$$

The formula for $\int F_i f(s)\,ds$ is given in (35), but a more explicit form for the E terms in that expression is now needed. From (28) can be obtained directly a formula for $\int F_i f(s)\,ds$. It is

$$\frac{\rho^{h_i}}{n\rho^{n-1}} \int_a^b F_i f(s)\,ds = -\sum_{j=\mu+1}^{n} W_{1i}(y_j) \int_a^b \bar{y}_j(s) f(s)\,ds$$

$$(37)\quad + \sum_{l=2}^{k-1} \left\{ \begin{matrix} -\sum_{j=\mu+1}^{n} W_{li}(y_j) \int_a^b \bar{y}_j(s) f(s)\,ds \\ \sum_{j=1}^{\mu} W_{li}(y_j) \int_a^b \bar{y}_j(s) f(s)\,ds \end{matrix} \right\} + \sum_{j=1}^{\mu} W_{ki}(y_j) \int_a^b \bar{y}_j(s) f(s)\,ds.$$

By integrating by parts ν times,

$$\int_\sigma^\tau e^{-\rho w_j(s-a)} f(s)\,ds = \frac{e^{-\rho w_j(\tau-a)}}{-\rho w_j}[f(\tau)] + \frac{e^{-\rho w_j(\sigma-a)}}{\rho w_j}[f(\sigma)] + \frac{E}{\rho^{\nu+1}},$$

where

$$[f(\tau)] \equiv f(\tau) + \frac{f'(\tau)}{\rho w_j} + \cdots + \frac{f^{(\nu-1)}(\tau)}{(\rho w_j)^{\nu-1}},$$

and $[f(\sigma)]$ has the same form. And now this can be applied to each term of $\bar{y}_j(s)$ in (26) with the result

$$\frac{n\rho^{n-1}}{w_j} \int_\sigma^\tau \bar{y}_j(s) f(s)\,ds = \frac{e^{-\rho w_j(\tau-a)}}{-\rho w_j}[h(\tau)] + \frac{e^{-\rho w_j(\sigma-a)}}{\rho w_j}[h(\sigma)] + \frac{E}{\rho^{\nu+1}},$$

where the symbol $[h(\tau)]$ has the more explicit form

$$[h(\tau)] = h(\tau) + \frac{h_1(\tau)}{\rho w_j} + \frac{h_2(\tau)}{(\rho w_j)^2} + \cdots + \frac{h_{\nu-1}(\tau)}{(\rho w_j)^{\nu-1}},$$

and $[h(\sigma)]$ is the same with the value σ substituted in place of τ. The function h is the function f, but a different notation is introduced because

the functions h_r that enter are not precisely the derivatives of f. The essential thing in the expression is the fact that the functions h_r are independent of j. When this result is used in (37) and the asymptotic solutions are substituted, the result must be of the form (35). Then by comparing the two forms the more explicit form for the E terms in (35) is found:

$$\int F_i f(s)\,ds \equiv \sum_{l=2}^{k-1}\left(\sum_{j=1}^{\mu} w_j^{h_{li}}[\alpha_{li}][h(a)]\rho^{h_{li}-h_i-1}e^{\rho w_j(a_l-a)} + \sum_{j=\mu+1}^{n} w_j^{h_{li}}[\bar{\alpha}_{li}][h(b)]\rho^{h_{li}-h_i-1}e^{\rho w_j(a_l-a)}\right)+\frac{E}{\rho}, \tag{38}$$

in which $[\alpha_{li}]$ is identical with the $[\alpha_{li}]$ appearing in D_i (i. e., when the bracket symbol is written out in full for any value of m, all corresponding terms as well as the E terms are identical). In the same way $[\bar{\alpha}_{li}]$ is identical with the $[\alpha_{li}]$ appearing in $\bar{H}_i$. This evaluation of the integrals is now substituted in (36), and each determinant there can be written as the sum of several, according to the terms of (38). Since from (25) the expressions $D_i/\bar{\Delta}$ and $(H_i/\bar{\Delta})$ are bounded on γ_1, the determinants which involve the last column E/ρ are themselves of the form E/ρ, and that part of (36) is of the form (32), and so converges to zero as the radius of γ_1 becomes infinite. On the other hand the terms involving exponentials other than w_μ or $w_{\mu+1}$ reduce to the form (31), with one of the constants a_l $(l \neq 1, k)$ in place of x. Hence these also approach zero. All that is left is then

$$E_1 e^{\rho w_\mu(x-a)}\sum_{i,j}\frac{1}{\bar{\Delta}}\begin{vmatrix}\bar{H}_i & N_i\\ \bar{H}_j & N_j\end{vmatrix} + E_2 e^{\rho w_{\mu+1}(x-b)}\sum_{i,j}\frac{1}{\bar{\Delta}}\begin{vmatrix}D_i & N_i\\ D_j & N_j\end{vmatrix},$$

where

$$N_i \equiv \sum_{l=2}^{k-1} w_\mu^{h_{li}}[\alpha_{li}][h(a)]\rho^{h_{li}-h_i-1}e^{\rho w_\mu(a_l-a)} + \sum_{l=2}^{k-1} w_{\mu+1}^{h_{li}}[\bar{\alpha}_{li}][h(b)]\rho^{h_{li}-h_i-1}e^{\rho w_{\mu+1}(a_l-b)}.$$

By inequality (25) the expression $(N_i/\bar{\Delta})$ is of the form E/ρ. So when the values of D_i and $\bar{H}_i$ are substituted from (21a) the first two terms of each give expressions that reduce to the form (32) and so approach zero. What still remains is

$$\frac{E_1 e^{\rho w_\mu(x-a)}}{\bar{\Delta}}\sum_{i,j}\begin{vmatrix}A_{bi} & B_{ai}+B_{bi}\\ A_{bj} & B_{aj}+B_{bj}\end{vmatrix} + \frac{E_2 e^{\rho w_{\mu+1}(x-b)}}{\bar{\Delta}}\begin{vmatrix}A_{ai} & B_{ai}+B_{bi}\\ A_{aj} & B_{aj}+B_{bj}\end{vmatrix}, \tag{39}$$

where

$$A_{ai} \equiv \sum_{l=2}^{k-1}[\alpha_{li}]e^{\rho w_\mu(a_l-a)}(\rho w_\mu)^{h_{li}-h_i}w_\mu^{h_i},$$

$$A_{bi} \equiv \sum_{l=2}^{k-1} [\bar{\alpha}_{li}]\, e^{\rho w_\mu (b-a_l)} (-\rho w_\mu)^{h_{li}-h_i} (-w_\mu)^{h_i},$$

$$B_{ai} \equiv \sum_{l=2}^{k-1} [\alpha_{li}][h(a)]\, \rho^{h_{li}-h_{l-1}-1}\, e^{\rho w_\mu (a_l-a)}\, w_\mu^{h_{li}},$$

$$B_{bi} \equiv \sum_{l=2}^{k-1} [\bar{\alpha}_{li}][h(b)]\, \rho^{h_{li}-h_{l-1}-1}\, e^{\rho w_\mu (b-a_l)} (-w_\mu)^{h_{li}}.$$

These can be expanded into the sum of a large number of determinants. Such determinants as have the same exponential factor in both columns are zero, for example

$$\begin{vmatrix} [\bar{\alpha}_{li}]\, e^{\rho w_\mu (b-a_l)} (-\rho w_\mu)^{h_{li}-h_i} (-w_\mu)^{h_i} & w_\mu^{h_{li}} [\bar{\alpha}_{li}][h(b)]\, \rho^{h_{li}-h_{l-1}-1}\, e^{\rho w_\mu (b-a_2)} \\ [\bar{\alpha}_{lj}]\, e^{\rho w_\mu (b-a_l)} (-\rho w_\mu)^{h_{lj}-h_j} (-w_\mu)^{h_j} & w_\mu^{h_{lj}} [\bar{\alpha}_{lj}][h(b)]\, \rho^{h_{lj}-h_{j-1}-1}\, e^{\rho w_\mu (b-a_2)} \end{vmatrix},$$

for when the common factors

$$e^{2\rho w_\mu (b-a_l)} [h(b)] (-w_\mu)^{h_{li}} (w_\mu)^{h_{li}} \rho^{h_{li}-h_i} \rho^{h_{lj}-h_j} \frac{1}{\rho}$$

(it is to be noted that $[h(b)]$ is independent of the subscript) are removed, we have

$$\begin{vmatrix} [\bar{\alpha}_{li}] & [\bar{\alpha}_{li}] \\ [\bar{\alpha}_{lj}] & [\bar{\alpha}_{lj}] \end{vmatrix},$$

which is obviously zero, since the two terms in the same row of the determinant are identical.

Since $a_2 - a$ and $b - a_{k-1}$ are the shortest of the intervals ending at a or b, the terms that approach zero most slowly are $e^{\rho w_\mu (a_2-a)}$ and $e^{\rho w_\mu (b-a_{k-1})}$, and each of the other determinants in (39) can be written

$$E\rho^{2p-1}\, e^{\rho w_\mu (a_2-a)}\, e^{\rho w_\mu (b-a_{k-1})}.$$

This divided by $\bar{\Delta}$ can, by inequality (25), be written E/ρ, whence (39) is of the form (32), and so approaches zero as the radius of γ_1 becomes infinite.

To sum up,

$$I_{\gamma_1} = \tfrac{1}{2} f(x) + \epsilon$$

for the radius of γ_1 sufficiently large.

The treatment of the integral in S_2, that is over γ_2 is, step for step, parallel to the above, and the result is obtained that for the radius of γ_2 sufficiently large

$$I_{\gamma_2} = \tfrac{1}{2} f(x) + \epsilon.$$

By combining the two results, the convergence of (1) is established in the even order case.

A MEMOIR ON THE DOCTRINE OF ASSOCIATED FORMS*

BY

O. E. GLENN

The theory of associated forms, founded by Hermite,† and developed by Clebsch, and Sylvester, and by many others, has as its primary object the study of sets of concomitants of a system of forms $F_1, F_2, \cdots$, such that every concomitant of the system is quasi-reducible, on multiplication by a power of a certain concomitant of a set, in terms of the members of this set. By expansion of each form of a system of binary forms, in terms of a pair of linear covariants as arguments, there is obtained the so-called typical representation of the system in which all coefficients are invariants; and every concomitant of the system is rational in the associated set consisting of the totality of these invariants, the two argument forms, and a multiplier which is the resultant of the two arguments. Extension to typical representation of binary forms of even order in terms of three quadratic covariants as arguments has been made by Clebsch.‡

In the first section of the present paper is given a generalization of the above theory, in which the system of ground forms is represented typically in terms of two quadratic covariants, and, under restrictions, in terms of two *nics*. This generalization is both natural and comprehensive, and gives remarkable syzygetic relations among the forms of a complete system.

The second section is devoted to associated and to complete systems of concomitants of a binary form f under a general linear transformation t, which are functions of the coefficients of t as well as of the coefficients and variables of f. These concomitants have various unique properties; for instance, the factor r of a concomitant C in the equation expressing the invariant property, $C' = rC$, is not, in all cases, a power of the determinant of t.

In the third section one of a system of ground forms is assumed to be a universal covariant of a special group. If the usual domain of rationality of concomitants is then altered by adjunction of the multiplier invariant of a certain associated set (an absolute constant), this set furnishes a fundamental system under the special group. Certain obscurities in the theory of in-

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† Hermite, Journal für Mathematik, vol. 52 (1856), p. 1.

‡ *Theorie der binären-algebraischen Formen* (1872), p. 410.

variants and covariants of special transformations, which have persisted since the time of Boole, are made clear by thus identifying their theory with the general doctrine of associated forms.

I. Generalizations in the theory of associated forms

1. Formal groundwork of the theory. I showed a few years since that the binary form f of order $n(m+1)-1$, whose coefficients are arbitrary variables, has a unique expansion* in terms of any two binary $n$ics f_{1n}, f_{2n},

$$f_{1n} = \alpha_0 x_1^n + n\alpha_1 x_1^{n-1} x_2 + \cdots, \qquad f_{2n} = \beta_0 x_1^n + n\beta_1 x_1^{n-1} x_2 + \cdots,$$

provided only that the resultant R of f_{1n}, f_{2n} does not vanish, in the form

$$f = \sum_{i=0}^{m} \binom{m}{i} \phi_{in-1} f_{1n}^{m-i} f_{2n}^{i}. \tag{1}$$

The coefficients ϕ_{in-1} $(i = 0, \cdots, m)$ were determined as binary quantics of order $n-1$ in x_1, x_2, with coefficients linear in the coefficients $a_0, \cdots, a_{n(m+1)-1}$ of f and rational in the quantities $\alpha_0, \cdots, \alpha_n, \beta_0, \cdots, \beta_n$; integral also in the latter, save for a common denominator R^m. These conclusions conformed to the theory of a symbolical basis of a finite expansion, according to which f is represented symbolically as the mth power

$$f = (\Xi_1 f_{1n} + \Xi_2 f_{2n})^m = \Xi_{f_n}^m = \Xi'^m_{f_n},$$

and the quantics $\phi_{in-1} = \Xi_1^{m-i} \Xi_2^i$ $(i = 0, \cdots, m)$ are all of the same type, in the sense of covariant *types*; satisfying the equations

$$-\left(\alpha \frac{\partial}{\partial \beta}\right) \phi_{in-1} = (m-i)\, \phi_{i+1n-1},$$

$$-\left(\beta \frac{\partial}{\partial \alpha}\right) \phi_{in-1} = i\phi_{i-1n-1}; \qquad \left(\beta \frac{\partial}{\partial \alpha}\right) = \beta_0 \frac{\partial}{\partial \alpha_0} + \cdots + \beta_n \frac{\partial}{\partial \alpha_n}.$$

The infinitude of symbolical expressions

$$F = \sum k (\Xi\Xi')^p (\Xi\Xi'')^q \cdots \Xi'_{f_n} \Xi''^t_{f_n} \cdots,$$

constructed by analogy with the symbolical concomitants of a binary $m$ic, represent rational and integral isobaric functions of the forms $\phi_{in-1}, f_{1n}, f_{2n}$, and all F's are annihilated by the three isomorphic operators

$$(a) \qquad \Xi_2 \frac{\partial}{\partial \Xi_1} - f_{1n} \frac{\partial}{\partial f_{2n}},$$

$$(b) \quad m\phi_{1n-1} \frac{\partial}{\partial \phi_{0n-1}} + (m-1)\, \phi_{2n-1} \frac{\partial}{\partial \phi_{1n-1}} + \cdots + \phi_{mn-1} \frac{\partial}{\partial \phi_{m-1n-1}} - f_{1n} \frac{\partial}{\partial f_{2n}},$$

* These Transactions, vol. 15 (1914), p. 72.

$$(c) \qquad -\left(\alpha_0\frac{\partial}{\partial\beta_0}+\alpha_1\frac{\partial}{\partial\beta_1}+\cdots+\alpha_n\frac{\partial}{\partial\beta_n}\right),$$

and therefore by $[\beta(\partial/\partial\alpha)]$ also.* This is a sufficient as well as a necessary condition for the invariance of the F's under the formal alterations of the quantics f_{1n}, f_{2n}, given by the equalities

$$f'_{1n}=\lambda_1 f_{1n}+\mu_1 f_{2n}, \qquad f'_{2n}=\lambda_2 f_{1n}+\mu_2 f_{2n},$$

where $(\lambda\mu)\neq 0; f'_{1n}=\alpha'_0 x_1^n+n\alpha'_1 x_1^{n-1}x_2+\cdots$.

In this paper much use is made of the explicit expressions for the forms ϕ_{in-1} in terms of the α's and β's. If we assume

$$\phi_{in-1}=p_{i0}x_1^{n-1}+p_{i1}x_1^{n-2}x_2+\cdots+p_{in-1}x_2^{n-1} \quad (i=0,\cdots,m),$$

and regard equation (1) as an identity, it gives just a sufficient number of linear equations in the quantities p_{ij} for their unique determination by Cramer's rule. The absolute terms of this linear system are respectively the coefficients $a_0, a_1, \cdots, a_{n(m+1)-1}$ of f. The matrix of the system is the matrix of the coefficients of the $n(m+1)$ forms of order $n(m+1)-1$ below, save for binomial multipliers common to elements of columns, which we omit (cf. (1)):

$$x_1^i x_2^j f_{1n}^m,\ x_1^i x_2^j f_{1n}^{m-1} f_{2n},\ \cdots,\ x_1^i x_2^j f_{2n}^m \qquad (j=0,\cdots,n-1;\ i=n-j-1).$$

The determinant of this matrix is a constant times $R^{\frac{1}{2}m(m+1)}\neq 0$.

Example. If $n=2$ and f is any form of odd order $2m+1$, the matrix is

(2)

p_{00}	p_{01}	$\cdots$	p_{i0}	p_{i1}	$\cdots$	p_{m1}	
α_0^m	0	$\cdots$	$\alpha_0^{m-i}\beta_0^i$	0	$\cdots$	0	a_0
$\dfrac{\delta\alpha_0^m}{\lfloor 1}$	α_0^m	$\cdots$	$\dfrac{\delta\alpha_0^{m-i}\beta_0^i}{\lfloor 1}$	$\alpha_0^{m-i}\beta_0^i$	$\cdots$	β_0^m	a_1
. . . . .	. . . . .		. . . . .	. . . . .		. . . . .	.
. . . . .	. . . . .		. . . . .	. . . . .		. . . . .	.
$\dfrac{\delta^{mn}\alpha_0^m}{\lfloor mn}$	$\dfrac{\delta^{mn-1}\alpha_0^m}{\lfloor mn-1}$	$\cdots$	$\dfrac{\delta^{mn}\alpha_0^{m-i}\beta_0^i}{\lfloor mn}$	$\dfrac{\delta^{mn-1}\alpha_0^{m-i}\beta_0^i}{\lfloor mn-1}$	$\cdots$	$\dfrac{\delta^{mn-1}\beta_0^m}{\lfloor mn-1}$	a_{2m}
0	$\dfrac{\delta^{mn}\alpha_0^m}{\lfloor mn}$	$\cdots$	0	$\dfrac{\delta^{mn}\alpha_0^{m-i}\beta_0^i}{\lfloor mn}$	$\cdots$	$\dfrac{\delta^{mn}\beta_0^m}{\lfloor mn}$	a_{2m+1}

where the column on the right is the column of absolute terms, and

$$\delta=2\alpha_1\frac{\partial}{\partial\alpha_0}+\alpha_2\frac{\partial}{\partial\alpha_1}+2\beta_1\frac{\partial}{\partial\beta_0}+\beta_2\frac{\partial}{\partial\beta_1}.$$

* White, American Journal of Mathematics, vol. 17 (1895), p. 235.

If we operate upon the determinant D of this matrix by either one of the operators

$$\left(\beta\frac{\partial}{\partial\alpha}\right),\ \left(\alpha\frac{\partial}{\partial\beta}\right),$$

the result is a sum of $2m+2$ determinants each having two columns alike and therefore zero. For, evidently,

$$\left(\beta\frac{\partial}{\partial\alpha}\right)f_{12}^{m-i}f_{22}^{i}=(m-i)f_{12}^{m-i-1}f_{22}^{i+1},$$

$$\left(\alpha\frac{\partial}{\partial\beta}\right)f_{12}^{m-i}f_{22}^{i}=if_{12}^{m-i+1}f_{22}^{i-1}\qquad(i=0,\cdots,m).$$

Hence D is a combinant of f_{12}, f_{22}, and, as the fundamental system of combinants of two quadratics contains only one invariant, their resultant R, we have $D=kR^{\frac{1}{2}m(m+1)}$, k being numerical.

For similar reasons the determinants of the augmented matrix corresponding to the solution by Cramer's rule for the respective quantities

$$p_{00},\ p_{10},\ \cdots,\ p_{r0},\ \cdots,\ p_{m0},$$

are all derivable from that corresponding to p_{00} by successive operation by $-[\alpha(\partial/\partial\beta)]$ upon the latter. Also, successive operation by $-[\alpha(\partial/\partial\beta)]$ upon the determinant of the augmented matrix corresponding to the solution for p_{01} gives the determinants of the augmented matrix corresponding to

$$p_{01},\ p_{11},\ \cdots,\ p_{i1},\ \cdots,\ p_{m1}.$$

A similar type of expansion of a binary form f_m of order m, with arbitrary coefficients, of which we shall make use, is one in terms of argument forms of unlike orders, f_{1n_1}, f_{2n_2}. We prove the following

LEMMA. *If f_{1n_1}, f_{2r_2} are any two given binary quantics of respective orders n_1, n_2, and if*

$$\phi_{im-n_i}=p_{i0}x_1^{m-n_i}+p_{i1}x_1^{m-n_i-1}x_2+\cdots+p_{im-n_i}x_2^{m-n_i}\quad(i=1,2),$$

then a necessary and sufficient condition in order that the linear system in the quantities p_{ij} obtained by making

$$f_m=\phi_{1m-n_1}f_{1n_1}+\phi_{2m-n_2}f_{2n_2}\qquad(n_1+n_2\leqq m+1),\tag{3}$$

identically, should be consistent, is that the resultant R of f_{1n_1} and f_{2n_2} should be different from zero.

To prove this we assume

$$f_{1n_1}=\alpha_0x_1^{n_1}+\alpha_1x_1^{n_1-1}x_2+\cdots,\qquad f_{2n_2}=\beta_0x_1^{n_2}+\beta_1x_1^{n_2-1}x_2+\cdots,$$

and, as $R \neq 0$ precludes $\alpha_0 = 0$, $\beta_0 = 0$ simultaneously, we let $\alpha_0 \neq 0$. The matrix of the linear system is now

$$M: \quad \begin{array}{c|cccccccc|c} & p_{10} & p_{11} & \cdots & p_{1m-n_1} & p_{20} & p_{21} & \cdots & p_{2m-n_2} & \\ \hline & \alpha_0 & 0 & \cdots & 0 & \beta_0 & 0 & \cdots & 0 & a_0 \\ & \alpha_1 & \alpha_0 & \cdots & 0 & \beta_1 & \beta_0 & \cdots & 0 & a_1 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \alpha_{n_1} & \alpha_{n_1-1} & \cdots & \cdots & \beta_{n_2} & \beta_{n_2-1} & \cdots & \cdots & \cdots \\ & 0 & \alpha_{n_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots & \alpha_{n_1} & 0 & 0 & \cdots & \beta_{n_2} & a_m \end{array}$$

If we delete the first $\sigma = m + 1 - n_1 - n_2$ columns of β's from M and apply to the resulting square matrix M' properly chosen elementary transformations we get a square matrix in which the first σ elements in the principal diagonal are α_0 and all other elements in the first σ rows and first σ columns are zero. Then the complementary minor of the minor of order σ in the upper left-hand corner of M' is the dialytic form of the resultant R, and, if D is the determinant of M', $D = \alpha_0^\sigma R$. Hence if $R \neq 0$ the rank of M is $m + 1$, and the lemma is proved.*

If $n_1 + n_2 = m + 1$, then $\sigma = 0$, and the expansion (3) is unique.

Again if the order $m - n_i$ of ϕ_{im-n_i} $(i = 1, 2)$ is sufficiently large, in fact if $m - n_i + 1 \geqq n_1 + n_2$, the form ϕ_{im-n_i} may itself be developed in an expansion (3) in terms of the argument forms f_{1n_1}, f_{2n_2},

$$\phi_{im-n_i} = \phi'_{1m-n_i-n_1} f_{1n_1} + \phi'_{2m-n_i-n_2} f_{2n_2},$$

and, so far as the orders permit repetitions, ν repetitions of this process for an arbitrary order m gives an expansion more explicit than (3), viz.,

$$(4) \qquad f_m = \phi_0^{(\nu)} f_{1n_1}^{\alpha} + \phi_1^{(\nu)} f_{1n_1}^{\alpha-1} f_{2n_2} + \cdots + \psi_1^{(\nu)} f_{1n_1} f_{2n_2}^{\beta-1} + \psi_0^{(\nu)} f_{2n_2}^{\beta},$$

wherein the coefficient forms are all of order less than $n_1 + n_2 - 1$. This expansion is not unique, in general, for, as seen above, arbitrary parameters remain in the coefficients of the forms $\phi^{(\nu)}$, $\psi^{(\nu)}$, except in particular cases analogous to the case $n_1 + n_2 = m + 1$ in (3).

2. Covariant expansions. Connection of the foregoing theory with invariant principles is made by regarding the argument forms in (1), (3), or (4) to be covariants of a definite set of linear transformations, either universal covariants of this set or covariants of f under the set. The following existence theorem proved for expansions of type (3) holds as well for type (1) and the unique cases of (4).

* Bôcher, *Introduction to Higher Algebra* (1915), p. 46.

THEOREM. *If f_{1n_1}, f_{2n_2} are covariants under a definite transformation or group G, and if $n_1 + n_2 = m + 1$ so that the expansion is unique, then the forms ϕ_{1m-n_1}, ϕ_{2m-n_2} are covariants of f under G.*

For proof let (3) be written in the form

$$(5) \quad f_m = \phi_1(A_0, A_1, \cdots; x_1, x_2) f_{1n_1} + \phi_2(\mathrm{A}_0, \mathrm{A}_1, \cdots; x_1, x_2) f_{2n_2},$$

where A_j is the rational function of $a_0, a_1, \cdots$ obtained by solving (3) for p_{1j} $(j = 0, \cdots, m - n_1)$ and A_j is the similar function equal to p_{2j} $(j = 0, \cdots, m - n_2)$. Let f_m be transformed by the general transformation of G, viz.,

$$t: \begin{cases} x_1 = \alpha_1 x_1' + \alpha_2 x_2', \\ x_2 = \beta_0 x_1' + \beta_1 x_2' \end{cases} \qquad (\alpha_1\beta_1 - \alpha_2\beta_0 \neq 0),$$

and suppose f_m' is expanded in terms of $f_{1n_1}(x_1', x_2')$, $f_{2n2}(x_1', x_2')$ by formula (3). Then

$$(6) \quad f_m' = \phi_1(A_0', A_1', \cdots; x_1', x_2') f_{1n_1}' + \phi_2(\mathrm{A}_0', \mathrm{A}_1', \cdots; x_1', x_2') f_{2n_2}',$$

where A_j', A_j' are the same functions of $a_0', a_1', \cdots$ that A_j, A_j respectively are of $a_0, a_1, \cdots$. Now if the inverse transformations on the x's and the a's are applied to all forms in equation (6) there results

$$(7) \quad f_m = \psi_1(B_0, B_1, \cdots; x_1, x_2) c_1 f_{1n_1} + \psi_2(\mathrm{B}_0, \mathrm{B}_1, \cdots; x_1, x_2) c_2 f_{2n_2},$$

in which c_1, c_2 are functions of the coefficients of the transformations t only, such that

$$f_i(x_1', x_2') = c_i f_i(x_1, x_2) \qquad (i = 1, 2).$$

Moreover $\psi_1 = \phi_1(A_0', \cdots)$, $\psi_2 = \phi_2(\mathrm{A}_0', \cdots)$ on account of the equalities of t. This expansion (7) must be identical with (3) since (3) is unique; and hence

$$(8) \quad \begin{aligned} \phi_1(A_1', A_0', \cdots; x_1', x_2') &= c_1^{-1} \phi_1(A_0, A_1, \cdots; x_1, x_2), \\ \phi_2(\mathrm{A}_0', \mathrm{A}_1', \cdots; x_1', x_2') &= c_2^{-1} \phi_2(\mathrm{A}_0, \mathrm{A}_1, \cdots; x_1, x_2), \end{aligned}$$

which proves the theorem.

For expansion (1) the equations analogous to (8) are

$$(9) \quad \phi_{in-1}(A_{i0}', A_{i1}', \cdots; x_1', x_2') = c_1^{i-m} c_2^{-i} \phi_{in-1}(A_{i0}, A_{i1}, \cdots; x_1, x_2) \qquad (i = 0, \cdots, m).$$

In the case $n_1 + n_2 < m + 1$ in (3) and for the non-unique cases of (4) the above proof does not hold. In numerous particular instances, however (cf. § III (26)), I have found that the arbitrary parameters which remain in the coefficient forms ϕ in these expansions can be determined so that relations analogous to

$$\psi_1 = \phi_1(A_0', \cdots), \qquad \psi_2 = \phi_2(\mathrm{A}_0', \cdots),$$

hold. The functions $\phi_1 = \phi_{1m-n_1}$, $\phi_2 = \phi_{2m-n_2}$ are then covariants whose coefficients involve arbitrary parameters.

3. Generalized typical representation. Let S be any system of binary forms subject to general algebraic transformations t, and let K be a complete system of concomitants of S, which contains two covariants f_{1n}, f_{2n} of order n, whose resultant does not vanish.* Then every covariant C of S whose order is of the form $n(m+1)-1$ can be developed in a covariant expansion (1) and is thereby represented typically in terms of f_{1n}, f_{2n} as arguments, and of other concomitants of K of orders $< n$. The multiplier invariant in this representation is the resultant R of f_{1n}, f_{2n}.

If $n = 2$, C represents any covariant of S of odd order. Thus the concomitants in K of orders $\leqq 2$ form an associated system for the representation of all covariants of odd order, in which the multiplier invariant is the resultant of any pair of quadratic covariants whose resultant does not vanish.

If we multiply any covariant C of even order of S by a definite linear covariant l in K and expand lC in the arguments f_{12}, f_{22} by formula (1) we get an expression of C in terms of the covariants f_{12}, f_{22} and concomitants of orders < 2 in which the denominator concomitant is of the form lR^s.

Again, covariants of S can be represented typically in terms of two covariants of respective unlike orders n_1, n_2, when the orders of these covariants satisfy conditions which make expansion (3) or, more generally, expansion (4), unique. The multiplier invariant is then the resultant of the argument forms f_{1n_1}, f_{2n_2}.

The simplest system S affording numerous examples of this theory is the well-known system of a quadratic f and a cubic g. The fundamental set K here contains the following fifteen forms: Invariants $D = (f,f)^2$, $(f,\Delta)^2$, R, $(f^3,g^2)^6$, $(f^3,gQ)^6$, R being the discriminant of g; linear covariants $(f,g)^2$, $(f^2,g)^3$, $(f,Q)^2$, $(f^2,Q)^3$; quadratic covariants f, $\Delta = (g,g)^2$, $(f,\Delta)^1$; cubic covariants g, $(f,g)^1$, $Q = (g,\Delta)^1$.

The problem of determining the associated system A for orders $\leqq 2$ in this case requires only the selection of two of the three quadratic covariants as argument forms and expressing their resultant, the multiplier for the associated system, in terms of the invariants in K. All covariants in K of orders greater than 2 are then expressed in terms of A by expansion formula (1) (or (3) with $n_1 = n_2 = 2$).

The resultant ρ of any two quadratic forms is readily determined as the discriminant of their jacobian. Thus, writing $J = (\Delta,f)^1$,

$$\tfrac{1}{2}\rho(\Delta,f) = ((\Delta,f)^1,J)^2 = ((J,\Delta)^1,f)^2.$$

* For a proof that two quadratic covariants with non-vanishing resultant exist for every form of even order > 4 cf. Clebsch, loc. cit., p. 410.

Applying the Gordan series

$$\begin{bmatrix} \Delta & f & \Delta \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix},$$

we have

$$(J, \Delta)^1 = \tfrac{1}{2}Rf - \tfrac{1}{2}\Delta\,(f, \Delta)^2.$$

Hence,

$$\tfrac{1}{2}\rho\,(\Delta, f) = \tfrac{1}{2}RD - \tfrac{1}{2}[\,(f, \Delta)^2]^2.$$

Similarly,

$$\tfrac{1}{2}\rho\,((f, \Delta)^1, f) = \tfrac{1}{4}RD^2 - \tfrac{1}{4}D[\,(f, \Delta)^2]^2,$$

$$\tfrac{1}{2}\rho\,((f, \Delta)^1, \Delta) = \tfrac{1}{4}R^2 D - \tfrac{1}{4}R[\,(f, \Delta)^2]^2.$$

We append the resulting representation of the covariant g when the arguments are chosen to be Δ, f:

$$g = \frac{\tfrac{1}{2}[\,(f^2, Q)^3 + \tfrac{1}{2}(f, g)^2\,(f, \Delta)^2]\,\Delta - \tfrac{1}{2}R\,(f, g)^2 f}{-\{RD - [\,(f, \Delta)^2]^2\}}. \tag{10}$$

In the case where S consists of a single binary quintic f the fundamental system K consists of 23 concomitants, of which three are quadratics, viz.,

$$i = (f, f)^4,\ (i^2, H)^4,\ (i^3, H)^5,$$

where H is the hessian of f. Selecting the first two of these as argument forms, all covariants in K of odd order > 2, of which there are eight, are represented in terms of an associated set of orders $\leqq 2$. The hermitian skew invariant does not occur in the multiplier invariant ρ as its degree is 18, whereas the degree of ρ is 16.

II. Systems belonging to prescribed domains

1. Domains of rationality. The general transformation t leaves fixed a point (ka, kb) provided that

$$s = (\alpha_1 s + \alpha_2)/(\beta_0 s + \beta_1) \qquad (s = a/b).$$

Thus t leaves invariant the two linear forms

$$f_1 = 2\beta_0 x_1 + (\beta_1 - \alpha_1 + \Delta)x_2, \qquad f_{-1} = 2\beta_0 x_1 + (\beta_1 - \alpha_1 - \Delta)x_2,$$

where $\Delta = \sqrt{(\beta_1 - \alpha_1)^2 + 4\beta_0\alpha_2}$, the roots of f_1, f_{-1} being the poles of t. Also t has the covariant $f_1 f_{-1}$, or

$$J = \beta_0 x_1^2 + (\beta_1 - \alpha_1)x_1 x_2 - \alpha_2 x_2^2.$$

Ordinary rational integral concomitants of a form under t belong to a domain, which we designate by $R_3(1, 0, 0)$, determined by the coefficients and variables of that form. The covariant J belongs to a domain obtained

from R_3 by adjunction of the coefficients of the transformation t, which will be indicated by $R_2(1, t, 0)$. Moreover, f_1 and f_{-1} belong to a domain $R_1(1, t, \Delta)$ obtained by adjunction of Δ to R_2.

The forms f_1, f_{-1}, J, and the concomitants considered below in this section, being functions of the coefficients of t, are primarily invariants of a single transformation t, but α_1 and β_1 appear in them only in the combination $\alpha_1 - \beta_1$; hence, if ϵ is a fixed constant, our invariants belong to a one-parameter continuous group leaving two points fixed, and characterized by the equation $\alpha_1 - \beta_1 = 2\epsilon\beta_0$. An important sub-problem under the general case is given by assuming

$$\alpha_1 - \beta_1 = 2\epsilon\beta_0, \qquad \alpha_2 = -\beta_0.$$

Then J, deprived of a constant factor, is a universal covariant of the set given by

$$t_1: \begin{aligned} x_1 &= (2\epsilon\beta_0 + \beta_1)x_1' - \beta_0 x_2', \\ x_2 &= \beta_0 x_1' + \beta_1 x_2', \end{aligned}$$

of which the orthogonal transformations are a particular case. All of our systems, particularized for t_1, belong essentially to the domain $R_3(1, 0, 0)$.

Both orthogonal and boolean systems are special cases of the general systems given in this section. We shall revert to this phase of the question in Section III.

2. Complete systems in $R_1(1, t, \Delta)$ of the transformation t. If a binary form $f = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \cdots$ be represented typically in terms of f_1, f_{-1} as argument forms, the multiplier invariant, a power of which occurs in the denominator of the expansion of f, is $4\beta_0\Delta$. This is an absolute constant and is, essentially, the quantity adjoined to define $R_1(1, t, \Delta)$. Writing the expansion in the form

$$(11) \qquad f = \phi_m f_1^m + m\phi_{m-2} f_1^{m-1} f_{-1} + \cdots + m\phi_{-(m-2)} f_1 f_{-1}^{m-1} + \phi_{-m} f_{-1}^m,$$

we have in ϕ_{m-2i} $(i = 0, \cdots, m)$ a set of $m + 1$ linearly independent invariants of t belonging to R_1, linear in the coefficients $a_0, \cdots, a_m$. The factor r of ϕ_{m-2i} in the equation expressing the invariant property

$$\phi'_{m-2i} = r\phi_{m-2i},$$

is not a power of the determinant of t, $(\alpha_1\beta_1 - \alpha_2\beta_0)$, but is a product of powers of the factors ρ_1, ρ_{-1} of this determinant in R_1,

$$(12) \qquad \rho_1 = \tfrac{1}{2}(\alpha_1 + \beta_1 + \Delta), \qquad \rho_{-1} = \tfrac{1}{2}(\alpha_1 + \beta_1 - \Delta).$$

In fact, writing $c_1 = \rho_1^{-1}$, $c_2 = \rho_{-1}^{-1}$, it results from I (9) that

$$(13) \qquad \begin{aligned} \phi'_{m-2i} &= c_1^{i-m} c_2^{-i} \phi_{m-2i} \\ &= \rho_1^{m-2i} (\alpha_1\beta_1 - \alpha_2\beta_0)^i \phi_{m-2i} \qquad (i = 0, \cdots, m). \end{aligned}$$

Moreover there follows the

THEOREM. *The $m+3$ forms f_1, f_{-1}, ϕ_{m-2i} $(i = 0, \cdots\cdots, m)$ of the associated set constitute a fundamental system of concomitants of f under t in $R_1(1, t, \Delta)$.*

The explicit systems in R_1 for $m = 1, 2, 3$ are given below, in which, to secure isobarism, we write $\beta_1 - \alpha_1 = \gamma_1$.

TABLE

$$m = 1$$

$$2^2\beta_0\Delta\phi_1 = (-\gamma_1 + \Delta)a_0 + 2\beta_0 a_1,$$

$$-2^2\beta_0\Delta\phi_{-1} = (-\gamma_1 - \Delta)a_0 + 2\beta_0 a_1.$$

$$m = 2$$

$$2^3\beta_0^2\Delta^2\phi_2 = (\gamma_1^2 + 2\alpha_2\beta_0 - \gamma_1\Delta)a_0 + 2\beta_0(-\gamma_1 + \Delta)a_1 + 2\beta_0^2 a_2,$$

$$2\beta_0\Delta^2\phi_0 = \alpha_2 a_0 + \gamma_1 a_1 - \beta_0 a_2,$$

$$2^3\beta_0^2\Delta^2\phi_{-2} = (\gamma_1^2 + 2\alpha_2\beta_0 + \gamma_1\Delta)a_0 + 2\beta_0(-\gamma_1 - \Delta)a_1 + 2\beta_0^2 a_2.$$

$$m = 3$$

$$-2^4\beta_0^3\Delta^3\phi_3 = (\gamma_1^3 + 3\beta_0\gamma_1\alpha_2 - (\gamma_1^2 + \beta_0\alpha_2)\Delta)a_0 + 3\beta_0(-\gamma_1^2 - 2\beta_0\alpha_2 + \gamma_1\Delta)a_1 + 3\beta_0^2(\gamma_1 - \Delta)a_2 - 2\beta_0^3 a_3,$$

$$-2^4\beta_0^3\Delta^3\phi_1 = \beta_0\alpha_2(\gamma_1 - \Delta)a_0 + \beta_0(\gamma_1^2 - 2\beta_0\alpha_2 - \gamma_1\Delta)a_1 + \beta_0^2(-3\gamma_1 + \Delta)a_2 + 2\beta_0^3 a_3,$$

$$2^4\beta_0^3\Delta^3\phi_{-1} = \beta_0\alpha_2(\gamma_1 + \Delta)a_0 + \beta_0(\gamma_1^2 - 2\beta_0\alpha_2 + \gamma_1\Delta)a_1 + \beta_0^2(-3\gamma_1 - \Delta)a_2 + 2\beta_0^3 a_3,$$

$$2^4\beta_0^3\Delta^3\phi_{-3} = (\gamma_1^3 + 3\beta_0\gamma_1\alpha_2 + (\gamma_1^2 + \beta_0\alpha_2)\Delta)a_0 + 3\beta_0(-\gamma_1^2 - 2\beta_0\alpha_2 - \gamma_1\Delta)a_1 + 3\beta_0^2(\gamma_1 + \Delta)a_2 - 2\beta_0^3 a_3.$$

3. Systems belonging to $R_2(1, t, 0)$. It will be shown hereafter that complete systems of f in R_2 may be constructed from associated systems by taking the set S of ground forms to be the quantics f, J. Since, however, the forms ϕ_{m-2i} do not satisfy some of the hypotheses (cf. (13)) fulfilled by ordinary invariants of f under the group of general linear algebraic transformations, it is desirable to develop a special method for systems of f under t in R_2, and later to identify the results obtained by the two methods to be employed.

Concomitants in R_1 or R_2 may evidently be constructed by forming linear expressions ψ in products of powers of f_1, f_{-1}, ϕ_{m-2i} $(i = 0, \cdots, m)$.* A necessary, though not sufficient, condition in order that such an expression may be an invariant function in R_2 is that the exponent ν of ρ_1 (cf. (13)) in each and every term of the equation expressing invariancy should be 0. The exponent ν for every term of any concomitant is equal to the sum of the

*Cf. Elliott, *The syzygetic theory of orthogonal binariants*, Proceedings of the London Mathematical Society, vol. 33 (1901), p. 226.

subscripts of the ϕ's diminished by the exponent of f_1 and increased by that of f_{-1} in any term. For pure invariants therefore,

$$\nu = \sum_i (m - 2i).$$

If d is the degree of such an invariant ψ and if we revert to the notation for the ϕ's used in I (1), $\phi_m = \phi_{00}$, $\phi_{m-2} = \phi_{10}$, $\cdots$, the sum w of the (first) subscripts in each term of ψ is $\sum i$. Thus

$$\nu = dm - 2w, \qquad w = \tfrac{1}{2}(dm - \nu).$$

We have thus established a one-to-one correspondence between invariants in the ϕ's, of degree d and factor ρ_1^ν, and isobaric invariant functions of degree d and weight w. The number of invariant expressions ψ with factor ρ_1^ν is therefore the number of partitions of w denoted by

$$(\tfrac{1}{2}(dm - \nu); d, m) = (\tfrac{1}{2}(dm + \nu); d, m).$$

Now in the domain $R_2(1, t, 0)$ we are concerned with the case $\nu = 0$. The actual invariant products of the ϕ's for which $\nu = 0$, each affected by that power of D which is the multiplier in the invariant relation for the product, may be found from the caylean real generating function

$$G(f) = \left\{ \prod_{i=0}^{m} (1 - \phi_{m-2i}\, \rho_1^{m-2i}\, D^i) \right\}^{-1} \qquad (D = \alpha_1 \beta_1 - \alpha_2 \beta_0). \tag{14}$$

The expansion of this function in ascending powers of ρ_1 has for the terms free from ρ_1 in it the sum of all of the products of the ϕ's which are invariants with $\nu = 0$, each product affected by the power of D which is its multiplier. The invariants of this set which are asyzygetic are those possessing the same power of D as multiplier.

Not all of these invariants belong to R_2. But the products may be arranged in conjugate pairs (ψ_p, ψ_{-p}), etc., and by replacing ψ_p, ψ_{-p} respectively by $\psi_p + \psi_{-p}$, $\psi_p - \psi_{-p}$ we get the complete asyzygetic sets which, neglecting factors involving Δ, belong to R_2. In fact ψ_{-p} is derived from ψ_p by changing all factors ϕ_{m-2i} into $\phi_{-(m-2i)}$; and that the power of D figuring as multiplier in the invariant relation for ψ_{-p} is the same as the power for ψ_p is easily proved. For, we have

$$\psi_p = \phi_{m-2i_1}^{p_1} \phi_{m-2i_2}^{p_2} \cdots, \qquad \psi_{-p} = \phi_{-(m-2i_1)}^{p_1} \phi_{-(m-2i_2)}^{p_2} \cdots.$$

The multiplier for ψ_p is $D^{p_1 i_1 + p_2 i_2 + \cdots}$, and that for ψ_{-p} is $D^{p_1(m-i_1) + p_2(m-i_2) + \cdots}$. But if $\nu = \sum_i (m - 2i) = 0$, then

$$p_1(m - i_1) + p_2(m - i_2) + \cdots = p_1 i_1 + p_2 i_2 + \cdots.$$

The generating function for simultaneous invariants is $\Pi G(f)$, the product

including the $G(f)$ function for each form involved, and covariants of a single form are simultaneous invariants of the form and a linear quantic. Hence the generating function for covariants of f in R_2 is

$$(15)\quad \Gamma(f) = \left\{ \prod_{i=0}^{m} (1 - \phi_{m-2i}\, \rho_1^{m-2i} D^i)(1 - f_1 \rho_1^{-1})(1 - f_{-1} \rho_1 D^{-1}) \right\}^{-1}.$$

Thus, we have the

THEOREM. *The products of functions ϕ_{m-2i} occurring in the portion of the expansion of $\Gamma(f)$ free from ρ_1, which are affected by the factor D^k, furnish a complete asyzygetic set of concomitants in R_2 of multiplier D^k. These products have only to be arranged in conjugate pairs (ψ_p, ψ_{-p}) and the terms of each pair replaced by $\psi_p + \psi_{-p}, \psi_p - \psi_{-p}$ respectively.*

We shall now illustrate this theorem by exhibiting the explicit systems given by these methods for the forms of orders 1, 2, 3.

$m = 1$. The generating function for invariants of a linear form f is

$$G(f) = \{(1 - \phi_1 \rho_1)(1 - \phi_{-1} \rho^{-1} D)\}^{-1}.$$

When this is expanded it is found that the terms free from ρ_1 are

$$1 + \phi_1 \phi_{-1} D + \phi_1^2 \phi_{-1}^2 D^2 + \cdots \equiv (1 - \phi_1 \phi_{-1} D)^{-1}.$$

Hence a linear form has only one invariant in R_2, viz., $\phi_1 \phi_{-1}$ of index 1. Moreover the terms free from ρ_1 in $G(f)$, when separated from the rest, are all given by the formula*

$$\frac{1}{1 - \phi_1 \phi_{-1} D}.$$

This formula will be called the real generating function for invariants in the domain R_2. The form of the denominator of such a function shows what invariants should be selected to form a complete system in this domain.

The generating function for covariants and invariants both is

$$\Gamma(f) = \{(1 - \phi_1 \rho_1)(1 - \phi_{-1} \rho_1^{-1} D)(1 - f_1 \rho_1^{-1})(1 - f_{-1} \rho_1 D^{-1})\}^{-1}.$$

Separation of the terms free from ρ_1 from $\Gamma(f)$ gives the terms furnished by the expansion of the following real generating function for concomitants of the linear form in R_2,

$$\frac{1 - \phi_1 \phi_{-1} f_1 f_{-1}}{(1 - \phi_1 \phi_{-1} D)(1 - \phi_1 f_1)(1 - \phi_{-1} f_{-1})(1 - f_1 f_{-1} D^{-1})}.$$

Accordingly the following four forms constitute a fundamental system of f under t in $R_2(1, t, 0)$ (cf. table for $m = 1$ in the preceding paragraph):

* Cf. Elliott, Proceedings of the London Mathematical Society, vol. 33 (1901), p. 236.

$$(16_1)\qquad \lambda = \phi_1\phi_{-1}, \qquad \lambda_1 = \phi_1 f_1 + \phi_{-1} f_{-1}, \qquad \lambda_{-1} = \phi_1 f_1 - \phi_{-1} f_{-1},$$
$$\mu = f_1 f_{-1},$$

where

$$I = 4\beta_0 \Delta^2 \lambda = \alpha_2 a_0^2 + \gamma_1 a_0 a_1 - \beta_0 a_1^2,$$

$$(16)\qquad C_{-1} = \lambda_{-1} = (-\gamma_1 a_0 + 2\beta_0 a_1) x_1 + (2\alpha_2 a_0 + \gamma_1 a_1) x_2,$$
$$C_1 = \lambda_1 = a_0 x_1 + a_1 x_2 \equiv f,$$
$$J = (4\beta_0)^{-1} \mu = \beta_0 x_1^2 + \gamma_1 x_1 x_2 - \alpha_2 x_2^2.$$

These forms are connected by the syzygy

$$S \equiv (4\beta_0 \alpha_2 + \gamma_1^2) f^2 - C_{-1}^2 - 4IJ = 0.$$

$m = 2$. The generating function for concomitants of a quadratic f is

$$\Gamma(f) = \{(1 - \phi_2 \rho_1^2)(1 - \phi_0 D)(1 - \phi_{-2} \rho_1^{-2} D^2)$$
$$(1 - f_1 \rho_1^{-1})(1 - f_{-1} \rho_1 D^{-1})\}^{-1}.$$

The real generating function for concomitants in R_2 is found to be

$$\frac{1 - \phi_2 \phi_{-2} f_1^2 f_{-1}^2}{(1 - \phi_0 D)(1 - \phi_2 \phi_{-2} D^2)(1 - \phi_2 f_1^2)(1 - \phi_{-2} f_{-1}^2)(1 - f_1 f_{-1} D^{-1})}.$$

Hence the fundamental system of f in R_2 consists of the following five forms from which extraneous factors involving Δ are to be deleted:

$$(17_1)\qquad \lambda_0 = \phi_0, \qquad \lambda = \phi_2\phi_{-2}, \qquad \lambda_1 = \phi_2 f_1^2 + \phi_{-2} f_{-1}^2,$$
$$\lambda_{-1} = \phi_2 f_1^2 - \phi_{-2} f_{-1}^2, \qquad \mu = f_1 f_{-1}.$$

This system is given in explicit form below:*

$$I_0 = 2\beta_0 \Delta^2 \lambda_0 = \alpha_2 a_0 + \gamma_1 a_1 - \beta_0 a_2,$$
$$I = 16\beta_0^2 \Delta^4 \lambda = \alpha_2^2 a_0^2 - 4\alpha_2 \beta_0 a_1^2 + \beta_0^2 a_2^2 + 2\alpha_2 \gamma_1 a_0 a_1$$
$$+ (\gamma_1^2 + 2\alpha_2 \beta_0) a_0 a_2 - 2\beta_0 \gamma_1 a_1 a_2,$$
$$C_1 = \Delta^2 \lambda_1 = ((\gamma_1^2 + 2\alpha_2 \beta_0) a_0 - 2\beta_0 \gamma_1 a_1 + 2\beta_0^2 a_2) x_1^2$$
$$(17)\qquad + 2(-\alpha_2 \gamma_1 a_0 + 4\alpha_2 \beta_0 a_1 + \beta_0 \gamma_1 a_2) x_1 x_2$$
$$+ (2\alpha_2^2 a_0 + 2\alpha_2 \gamma_1 a_1 + (\gamma_1^2 + 2\alpha_2 \beta_0) a_2) x_2^2,$$
$$C_{-1} = \Delta \lambda_{-1} = (-\gamma_1 a_0 + 2\beta_0 a_1) x_1^2 + 2(\alpha_2 a_0 + \beta_0 a_2) x_1 x_2$$
$$+ (2\alpha_2 a_1 + \gamma_1 a_2) x_2^2,$$
$$J = (4\beta_0)^{-1} \mu = \beta_0 x_1^2 + \gamma_1 x_1 x_2 - \alpha_2 x_2^2.$$

* In place of C_1 we may use f since $\Delta^2 f = C_1 + 2I_0 J$.

A single syzygy connects these forms, viz.,

$$S \equiv 4IJ^2 - C_1^2 + (4\beta_0\,\alpha_2 + \gamma_1^2)\,C_{-1}^2 = 0\,.$$

$m = 3$. In a similar manner, employing the real generating function for concomitants of the cubic in R_2, we derive as a fundamental system the following thirteen quantics (cf. table in II 2):

$$\begin{gathered}\phi_3\,\phi_{-3},\quad \phi_1\,\phi_{-1},\quad \phi_3\,\phi_{-1}^3 + \phi_{-3}\,\phi_1^3,\quad \phi_3\,\phi_{-1}^3 - \phi_{-3}\,\phi_1^3,\quad f_1 f_{-1},\\ \phi_1 f_1 \pm \phi_{-1} f_{-1},\qquad \phi_3 f_1^3 \pm \phi_{-3} f_{-1}^3,\\ \phi_3\,\phi_{-1} f_1^2 \pm \phi_{-3}\,\phi_1 f_{-1}^2,\qquad \phi_3\,\phi_{-1}^2 f_1 \pm \phi_{-3}\,\phi_1^2 f_{-1}.\end{gathered} \tag{18}$$

III. Invariant systems for special groups

1. Complete systems derived from associated systems. The systems treated in section II may be derived from certain associated systems, and by a method which is of importance in the invariant theory of various special sets of transformations.

Construct, in terms of the forms f_1, f_{-1} as arguments, the typical representation of the system S of ground forms consisting of an *mic* f and the covariant J. The representation of f is given by (11), that of J is J itself since $f_1 f_{-1} = 4\beta_0 J$. The multiplier of the associated system is the absolute constant $4\beta_0\,\Delta$ which we adjoin to the domain. The fundamental system of f under t in R_2 then consists of an associated system of $S = \{f, J\}$, with Δ, as a constant, omitted. We can construct this associated set explicitly by making the following substitution in the associated system of an *mic* f and a quadratic $J_1 = b_0 x_1^2 + 2b_1 x_1 x_2 + b_2 x_2^2$, in which the multiplier invariant is the square root of the discriminant of J_1:

$$\begin{pmatrix} a_i & b_0 & b_1 & b_2 & x_1 & x_2 \\ \phi_{m-2i} & 0 & 1 & 0 & f_1 & f_{-1} \end{pmatrix} \qquad (i = 0, \cdots, m). \tag{19}$$

Thus, in case $m = 2$, the covariant

$$(a_0 b_1 - a_1 b_0)\,x_1^2 + (a_0 b_2 - a_2 b_0)\,x_1 x_2 + (a_1 b_2 - a_2 b_1)\,x_2^2$$

becomes

$$\phi_2 f_1^2 - \phi_{-2} f_{-1}^2\,,$$

and this is the fourth form of (17_1). Similarly the sets (16_1), (17_1), (18) may all be derived from the corresponding associated systems of f and J_1.

2. Orthogonal systems. An obscurity which has long persisted in the theory of orthogonal invariants may now be cleared up. To quote from the paper by Elliott on orthogonal binariants referred to above: "Absolute orthogonal invariants of a p-ic are invariants of the p-ic and $x_1^2 + x_2^2$ and

vice versa. It has therefore been hastily supposed elsewhere that the complete irreducible invariant system of a p-ic and a quadratic produces exactly the complete (absolute) orthogonal system for the p-ic, when $x_1^2 + x_2^2$ is taken for the quadratic. A first case of the redundancy of the former system for the latter purpose is exhibited above [the reducibility of the covariant Q of the cubic in terms of the orthogonal system corresponding to the set (18)]. The search for complete absolute orthogonal systems is not identical with the search for invariant systems of forms one of which is a quadratic."

The inference here is that orthogonal systems require a special method of treatment not comprehended in the ordinary theory for integrally complete systems under the general binary group G. We can now show, however, that the orthogonal invariant theory is essentially identical with the theory of associated systems under G. For, by particularizing the transformation t_1 of II 1 into the orthogonal transformations

$$x_1 = x_1' \cos\theta - x_2' \sin\theta, \qquad x_2 = x_1' \sin\theta + x_2' \cos\theta$$

(making $\epsilon = 0$, $\beta_0 = \sin\theta$, $\beta_1 = \cos\theta$) J becomes essentially $x_1^2 + x_2^2$, while the concomitants exemplified in the table in II 2 are freed from the coefficients of the transformation and become the non-absolute orthogonal concomitants of the respective forms f, the multiplier Δ of the associated system now becoming a constant times $\sqrt{-1}$. At the same time the sets in R_2, (16_1), (17_1), (18) are also freed from the coefficients of the transformation and become the complete absolute orthogonal systems.

Thus the absolute orthogonal invariant system of a form f is given precisely by the associated system of the set f, $x_1^2 + x_2^2$, where the multiplier invariant is the square root of the discriminant of $x_1^2 + x_2^2$.

The lowest instance where this associated set is smaller in number than the ordinary integrally complete simultaneous system of f and $x_1^2 + x_2^2$ is where $m = 3$. In fact, in the general case of the system (18) of the transformation t, the covariant Q of the cubic is reducible, on multiplication by a power of Δ, in terms of the set (18).

3. Boolean systems. If we particularize t_1 of II 1 by the substitutions

$$\beta_0 = \frac{\sin\alpha}{\sin w}, \qquad \beta_1 = \frac{\sin\beta}{\sin w}, \qquad \epsilon = -\cos w, \tag{20}$$

and assume $w = \beta - \alpha$, we get the following transformations of determinant unity,

$$x_1 = x_1' \frac{\sin(w-\alpha)}{\sin w} + x_2' \frac{\sin(w-\beta)}{\sin w}, \qquad x_2 = x_1' \frac{\sin\alpha}{\sin w} + x_2' \frac{\sin\beta}{\sin w}, \tag{21}$$

representing rotation of a pair of oblique coördinate axes inclined at an angle w

into another pair inclined at the same angle. Then J becomes a constant times the universal covariant (quâ transformations on x_1, x_2 and not on w)

$$q = x_1^2 + 2\cos w x_1 x_2 + x_2^2.$$

Hence if we make the substitutions (20) in the systems (16_1), (17_1), (18) these become the boolean systems of the forms of respective orders 1, 2, 3.

Moreover, the invariants in the table in II 2 become, under (20), systems of a species of boolean invariant in which the factor in the invariant equation is a product of powers of ρ_1, ρ_{-1} as below (cf. (13)),

$$\rho_1 = \cos\alpha + i\sin\alpha, \qquad \rho_{-1} = \cos\alpha - i\sin\alpha.$$

The two invariants of the linear form, for example, become

$$(22) \qquad (\cos w - i\sin w)a_0 - a_1, \qquad (\cos w + i\sin w)a_0 - a_1.$$

4. Kleinian groups. The formal nature of the invariant systems treated in section II shows that they may be interpreted when the transformation t has complex coefficients α_1, α_2, β_0, β_1. In particular, the multiply infinite set represented by

$$t_1: \quad \begin{aligned} x_1 &= (2\epsilon\beta_0 + \beta_1)x_1' - \beta_0 x_2', \\ x_2 &= \beta_0 x_1' + \beta_1 x_2', \end{aligned}$$

where ϵ is a fixed and β_0, β_1 arbitrary complex numbers, has the universal covariant

$$q = x_1^2 - 2\epsilon x_1 x_2 + x_2^2;$$

and the systems tabulated in II, particularized for t_1, become complete systems with complex coefficients, free from the arbitrary coefficients in t_1. Note that the latter transformation includes all types of substitutions with complex coefficients excepting the parabolic type, i. e., includes the elliptic, hyperbolic, and the loxodromic types.*

5. The general cyclic group on n variables. The following transformations generate the so-called cyclic group C on n variables:

$$C: \quad (x) = \begin{bmatrix} \lambda_1, \lambda_2, \cdots, \lambda_n \\ \lambda_n, \lambda_1, \cdots, \lambda_{n-1} \\ \cdots\cdots\cdots \\ \lambda_2, \lambda_3, \cdots, \lambda_1 \end{bmatrix} (x').$$

The determinant D of C is a linearly factorable n-ary form of order n. For, let us multiply its columns by $1, g^t, \cdots, g^{(n-1)t}$, respectively, where g is an arbitrary primitive nth root of unity, and $t \leqq n-1$, and add all to the first

* Klein-Fricke, *Theorie der elliptischen Modulfunctionen* (1890), p. 163.

column. The result is immediately of the form

$$F_t(\lambda) \cdot \begin{vmatrix} 1, & \lambda_2, & \cdots, & \lambda_n \\ g^t, & \lambda_1, & \cdots, & \lambda_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ g^{(n-1)t}, & \lambda_3, & \cdots, & \lambda_1 \end{vmatrix},$$

where

$$F_t(\lambda) = \lambda_1 + g^t\lambda_2 + g^{2t}\lambda_3 + \cdots + g^{(n-1)t}\lambda_n;$$

and hence

$$D = \prod_{t=0}^{n-1} F_t(\lambda).$$

The following theorem is easily proved:

THEOREM. *The forms* $F_t(x)$ $(t = 0, \cdots, n-1)$ *are universal covariants of* C, *satisfying the invariant relations*

$$F_t(x') = F_{(n-1)t}(\lambda)^{-1} F_t(x). \tag{23}$$

The general quantic f of order m in n variables, whose coefficients are arbitrary variables, can be given a typical representation by expanding it in terms of the forms $F_t(x)$ $(t = 0, \cdots, n-1)$ as arguments. The multiplier of the resulting associated system, i. e., the invariant, a power of which occurs in the denominator in the expansion of f, is the eliminant of the n linear arguments.* This resultant, being a function of nth roots of unity only, is a constant which we may adjoin to the domain of rationality. The coefficients in the expansion will be linear expressions in the $\binom{m+n-1}{m}$ coefficients of f, belonging to the domain R of rational functions of the nth roots of unity. These linear expressions, $\binom{m+n-1}{m}$ in number, constitute a complete system of invariants of f under C in R.†

6. A transformation leaving an involution fixed. The systems in R_2 treated in section II are complete for any transformation t whose only covariant in R_2 is a power of the quadratic represented by J. There exist no covariants of the general transformation t other than $f_1 f_{-1}$ and its powers. This appears by consideration of the generating functions employed in II, but a direct proof of the fact is possible. Since t has but two poles, any other quadratic covariant of t, as

$$K = x_1^2 - 2bx_1x_2 + cx_2^2,$$

must have its roots interchanged by t. But the condition

$$b \mp \sqrt{b^2 - c} = \frac{\alpha_1(b \pm \sqrt{b^2 - c}) + \alpha_2}{\beta_0(b \pm \sqrt{b^2 - c}) + \beta_1} \tag{24}$$

gives $\beta_1 = -\alpha_1$.

* Compare a paper by the present writer, these Transactions, vol. 15 (1914), p. 83.

† For a treatment of the case $n = 3$, cf. Elliott, Messenger of Mathematics, vol. 33 (1903–4), p. 108.

Under the transformation with arbitrary coefficients,

$$T : x_1 = \alpha_1 x_1' + \alpha_2 x_2', \qquad x_2 = \beta_0 x_1' - \alpha_1 x_2',$$

the following quadratic forms are relative covariants:

$$f_{12} = \beta_0 x_1^2 + \alpha_2 x_2^2, \qquad f_{22} = \alpha_1 x_1^2 + \alpha_2 x_1 x_2.$$

These forms are left invariant by having their roots interchanged, and all quadratics thus left invariant belong to the involution

$$L = k_1 f_{12} + k_2 f_{22}.$$

In fact, equation (24) gives

$$\beta_0 c - 2\alpha_1 b - \alpha_2 = 0,$$

and hence the following equality is an identity:

$$K = \frac{2\alpha_1 b + \alpha_2}{\beta_0 \alpha_2} f_{12} - \frac{2b}{\alpha_2} f_{22}.$$

The covariant $J = \beta_0 x_1^2 - 2\alpha_1 x_1 x_2 - \alpha_2 x_2^2$ is the jacobian of f_{12}, f_{22}. Hence the

THEOREM. *A necessary and sufficient condition in order that a binary form be transformed into itself by T is that it be equal to a power of J times a product of powers of forms in the involution L. The double points of this involution are the poles of T.*

If we expand a quantic f_{2m+1} in the argument forms f_{12}, f_{22} according to the formula (1) I, the coefficient forms ϕ_{i1} $(i = 0, \cdots, m)$ are linear covariants of f_{2m+1} under T in $R_2(1, T, 0)$. Then it is manifest that a fundamental system of concomitants of f_{2m+1} under T in this domain is precisely the simultaneous system of the $m + 1$ linear covariants ϕ_{i1} $(i = 0, \cdots, m)$ taken with the covariants of the transformation, e. g., f_{12}, f_{22}. Definitive processes appertaining to T for the construction of this joint system are not known.

We append the coefficient covariants ϕ_{01}, ϕ_{11} for the case where the expanded form is a cubic. If $D = -\alpha_1^2 - \beta_0 \alpha_2$, then,

$$-\alpha_2 D\phi_{01} = (\alpha_2^2 a_0 - 3\alpha_1 \alpha_2 a_1 + 3\alpha_1^2 a_2 + \alpha_1 \beta_0 a_3) x_1 - D a_3 x_2,$$

$$(25) \quad -\alpha_2 D\phi_{11} = (\alpha_1 \alpha_2 a_0 + 3\alpha_2 \beta_0 a_1 - 3\alpha_1 \beta_0 a_2 - \beta_0^2 a_3) x_1 + (-\alpha_2^2 a_0 + 3\alpha_1 \alpha_2 a_1 + 3\alpha_2 \beta_0 a_2 - \alpha_1 \beta_0 a_3) x_2.$$

Note that the so-called skew-orthogonal transformations form a special case of T, but, also, that in this case the poles are real and rational, and the domains R_1, R_2 coincide.

7. Modular systems. One of the main features of the general theory of

sections I, II of this paper is its character as a formal algorithm under which a variety of invariant systems for special groups can be studied.

Consider the group of binary transformations T in which the four parameters are least residues modulo p, where p is a prime number. A complete system of universal covariants mod p of T consists of*

$$L = x_1^p x_2 - x_1 x_2^p, \qquad Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2}) \div L = x_1^{p(p-1)} + \cdots + x_2^{p(p-1)}.$$

Regarding all equations in I 2 to be congruences modulo p, and setting $f_{1n_1} = Q$, $f_{2n_2} = L$ in (3) and in (4), it appears that a binary form of order $m = p^2$ is reducible in terms of Q and L and two first degree formal modular covariants of respective orders p and $p^2 - p - 1$. This raises the question whether the modular expansions (3) for $m > p^2$, containing arbitrary parameters in their coefficients, may always have these parameters determined so the coefficient forms ϕ_{1m-p^2+p}, ϕ_{2m-p-1} are modular covariants. Assuming this, (4) shows that the general quantic of order $m > p^2 - 1$ is reducible modulo p in terms of Q, L, and its own first degree covariants of orders from 0 to $p^2 - 1$ inclusive.

For the case $p = 2$ I have answered this question in the affirmative for the general order m.†

For the case $p = 3$ I have actually reduced in this manner the forms of orders $m = 3^2$, $m = 10$, $m = 11$, thus verifying the existence of the desired expansion in a very typical case. Thus, while the theorem given below has not been proved in as general a form as is stated, I venture to give it in its general form for the notice of mathematicians who may be interested.

Since the proposition relates to the reducibility of the general quantic we may state it as a theorem on covariants.

THEOREM. *The maximum order of an irreducible covariant modulo p of any binary form or set of forms is $p^2 - 1$.*

For the case $m = 11$, $p = 3$, we find $f \equiv \phi_{15} Q + \phi_{27} L \pmod 3$, where

$$\begin{aligned}\phi_{15} = {} & a_0 x_1^5 + (a_1 + a_3 + \lambda S) x_1^4 x_2 + (a_2 + a_4 + a_6 - \lambda T) x_1^3 x_2^2 \\ & + (a_5 + a_7 + a_9 - \lambda S) x_1^2 x_2^3 + (a_8 + a_{10} + \lambda T) x_1 x_2^4 + a_{11} x_2^5,\end{aligned} \tag{26}$$

where

$$S = a_1 + a_3 + a_5 + a_7 + a_9, \qquad T = a_2 + a_4 + a_6 + a_8 + a_{10},$$

and λ is any least positive residue modulo 3. The septic covariant ϕ_{27} is readily obtained as the quotient modulo 3 of $f - Q\phi_{15}$ divided by L.

A complete system of formal modular concomitants of a binary form f could be obtained, under this theorem, as a simultaneous system of a set of

* Dickson, these Transactions, vol. 12 (1911), p. 75.

† O. E. Glenn, these Transactions, vol. 17 (1916), p. 545.

first degree covariants of f of orders not exceeding $p^2 - 1$. Note, however, that the methods by which this simultaneous system would be constructed would be the definitive construction processes for simultaneous invariants and covariants under the modular group T, and these definitive construction processes are not fully known. Nevertheless, for fundamental systems of pure invariants in the modular realm, Dickson's theory of classes furnishes the requisite definitive methods.

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THE DEGREE AND CLASS OF MULTIPLY TRANSITIVE GROUPS*

BY

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If a group contains substitutions of degree μ but none that displace fewer letters, the identity excepted, μ is said to be its class.† The same term is unfortunately used with a different meaning in the theory of prime power groups.

In substitution group theory the two problems of the order and of the class of groups of given degree are in large measure interchangeable.‡ For this reason Bochert's inequalities, limiting the degree n of a multiply transitive group in terms of its class μ are second in importance only to Sylow's theorem.§ Bochert's theorem may be stated as follows.||

The class (>3) *of a substitution group of degree* n *exceeds* $\frac{1}{3}n - \frac{2}{3}\sqrt{n}$ *if it is doubly,* $\frac{1}{3}n - 1$ *if triply,* $\frac{1}{2}n - 2$ *if quadruply, transitive.*

In the proof of this theorem Bochert introduces the condition that the class is a given number by means of the commutator of two non-commutative substitutions. Now the commutator is only one combination out of many that may be made from the substitutions of a group (G) and is by no means always one of the substitutions of lowest degree in G. In the hope of penetrating more deeply into this matter the author has assumed the presence of substitutions of order 2 among the substitutions of lowest degree in G and thereby gains the advantages that are afforded by the simplicity of structure of diedral rotation groups. The principal result arrived at may be put thus:

1. THEOREM. *Let* n *be the degree and* $\mu\ (>2)$ *the class of a group that contains a substitution of order* 2 *and degree* μ; *then if it is*

doubly transitive,		$\mu > \frac{1}{2}n - \frac{1}{2}\sqrt{n} - 1,$
triply	" ,	$\mu \geqq \frac{1}{2}n,$
4-*ply*	" ,	$\mu \geqq \frac{1}{2}(n+1),$
5-*ply*	" ,	$\mu \geqq \frac{3}{6}n,$
6-*ply*	" ,	$\mu > \frac{3}{4}n,$
7-*ply*	" ,	$\mu \geqq \frac{10}{18}n,$

* Presented to the Society, Nov. 25, 1916.

† Jordan, Paris Comptes Rendus, vol. 72 (1871), p. 384.

‡ See these Transactions, vol. 16 (1915), p. 139.

§ Jordan, Journal de mathématiques, ser. 5, vol. 1 (1895), p. 35.

|| Bochert, Mathematische Annalen, vol. 40 (1892), p. 176; vol. 49 (1897), p. 133.

more than $p_1 + p_2 + \cdots + p_r$ *times transitive, where* $p_1, p_2, \cdots, p_r$ *are distinct odd primes* $(r > 1)$,

$$\mu > \frac{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1) - 1}{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)} n.$$

The diedral groups

2. The auxiliary theorem by means of which this will be proved is the following:

If the order of a group of class μ *generated by two substitutions* s *and* t *of order* 2 *and degree* μ *is divisible by each of the odd primes* $p_1, p_2, \cdots, p_r$, *its degree* n *does not exceed* $\mu + \mu/(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)$; *if the order* N *of* st *is odd or is divisible by* 4, n *does not exceed* $\mu + 2\mu/N$; *if* N (> 2) *is twice an odd number,* n *is not greater than* $\mu + 2\mu/N + 8\mu/N^2$.

If we erase from $\{s, t\}$, of degree n, the ξ letters of all the regular constituents of $\{s, t\}$ which involve letters displaced by both s and t the group in the remaining constituents is of degree $n - \xi$, of class $\mu - \xi$, and is generated by two similar substitutions of order 2 and degree $\mu - \xi$. Then if the theorem can be shown to be true for diedral groups satisfying the given conditions, but in which the regular constituents, if any, are of order 2 and exhibited by transpositions of s (or t) in letters fixed by t (or s), it is *a fortiori* true when regular constituents on letters common to s and t are present. It may be noted that if p^a is a Sylow divisor of N, i. e., the highest power of the prime p that divides the order of st, and if p^a divides the degree of a regular constituent of $\{s, t\}$ it must also divide the degree of some non-regular constituent of $\{s, t\}$. If this is not so, $(st)^{N/p}$ involves no letter not in a regular constituent of $\{s, t\}$ and is not the identity. Thus the degree of $(st)^{N/p}$ is at least μ and cannot exceed ξ; but if $\xi = \mu$ the truth of the theorem is not in dispute. Consequently the deletion of the regular constituents from the group $\{s, t\}$ leaves a group of the same order. In what follows the group $\{s, t\}$ under discussion will be assumed free from any regular constituent on letters common to s and t.

The transitive constituents of $\{s, t\}$ may be enumerated as follows. In s there are m_1 cycles that displace letters not in t, and t has m_2 cycles that displace letters not in s; let $m = m_1 + m_2$. There are y_j constituents of degree Y_j and order $2Y_j$, Y_j an odd number, $j = 1, 2, \cdots$. There are z_k' transitive constituents of degree Z_k and order $2Z_k$, Z_k even, $k = 1, 2, \cdots$, with the generator of degree Z_k in s and the generator of degree $Z_k - 2$ in t. In like manner there are z_k'' constituents of the same order Z_k with $Z_k - 2$ letters in s and Z_k letters in t; let $z_k = z_k' + z_k''$. The numbers $Y_1, Y_2, \cdots$ are all the odd divisors of N, unity excepted, and the numbers $Z_1, Z_2, \cdots$

are all the even divisors of N, with the exception of 2. We recall that a non-regular transitive diedral rotation group of degree n is of class $n-1$ if n is odd, and is of class $n-2$ if n is even. In the latter case the two generators of order 2 are not similar substitutions. Of course some of the numbers $y_1, y_2, \cdots, z_1', z_2', \cdots, z_1'', z_2'', \cdots$ may be zero. The degree of s is

$$2m_1 + \sum y_j(Y_j - 1) + \sum z_k' Z_k + \sum z_k''(Z_k - 2),$$

and the degree of t is

$$2m_2 + \sum y_j(Y_j - 1) + \sum z_k'(Z_k - 2) + \sum z_k'' Z_k.$$

The summations extend the subscripts through their entire range of values. With this understanding we may omit the subscripts j and k in future and introduce these abbreviations:

$$E_1 \equiv \sum y(Y-1), \qquad F_1 \equiv \sum z(Z-1),$$

and similarly

$$\Gamma_a \equiv E_a + F_a \equiv \sum y(Y-a) + \sum z(Z-a).$$

Then the degree of $\{s, t\}$ is $2m + \Gamma_0$ which on the elimination of $2m$ becomes $2\mu - \Gamma_2$; for

$$-2\Gamma_1 + \Gamma_0 = \sum yY - \sum(2yY - 2y) + \sum zZ - \sum(2zZ - 2z)$$
$$= -\sum y(Y-2) - \sum z(Z-2) = -\Gamma_2.$$

3. If st is of odd prime order it is obvious that $n = \mu + \mu/(p-1)$.

Let st be of order p^α, p *any* prime and $\alpha > 1$. Let $\{s, t\}$ have x_i transitive constituents of degree p^i, $i = 1, 2, \cdots, \alpha$. From $(st)^{p^{\alpha-1}}$ we get the equation,

$$p^\alpha x_\alpha = \mu + h_\alpha,$$

from $(st)^{p^{\alpha-2}}$,

$$p^{\alpha-1} x_{\alpha-1} + p^\alpha x_\alpha = \mu + h_{\alpha-1},$$

from $(st)^{p^{\alpha-3}}$,

$$p^{\alpha-2} x_{\alpha-2} + p^{\alpha-1} x_{\alpha-1} + p^\alpha x_\alpha = \mu + h_{\alpha-2},$$

. . . . . .

from $(st)^p$,

$$p^2 x_2 + p^3 x_3 + \cdots + p^\alpha x_\alpha = \mu + h_2,$$

while

$$p x_1 + p^2 x_2 + \cdots + p^\alpha x_\alpha = \mu + \delta,$$

and

$$(p-1)x_1 + (p^2-1)x_2 + \cdots + (p^\alpha - 1)x_\alpha = \mu.$$

Here $h_2, h_3, \cdots, h_\alpha, \delta$, are positive integers or zero, and $\mu + \delta$ is the degree

of $\{s, t\}$. These $\alpha + 1$ equations are consistent, their eliminant must vanish, and when equated to zero and reduced it gives

$$\delta = \mu/p^{\alpha-1}(p-1) - \sum_{i=2}^{\alpha} h_i/p^{i-1}.$$

Then if st is of order p^α $(\alpha > 1)$, the degree of $\{s, t\}$ does not exceed

$$\mu + \mu/p^{\alpha-1}(p-1).$$

4. Let $N = 2p^\alpha$, where p^α is a power of an odd prime. Let y and z be the number of transitive constituents of degrees p^α and $2p^\alpha$ respectively in $\{s, t\}$. Then from $(st)^{N/p}$ and $(st)^{N/2}$ we have

$$p^\alpha y + 2p^\alpha z = \mu + h,$$

$$2m \qquad + 2p^\alpha z = \mu + k - \overline{F}_0,$$

and

$$2m + p^\alpha y + 2p^\alpha z = \mu + \delta - \overline{\Gamma}_0,$$

with

$$m + (p^\alpha - 1)y + (2p^\alpha - 1)z = \mu - \overline{\Gamma}_1.$$

By $\overline{F}_0$, $\overline{\Gamma}_0$, and Γ_1 is meant the summation explained above but with the exclusion of constituents whose degree is divisible by p^α. The meaning of h, k, and δ is the same as before. They are positive integers or zero. The eliminant of m, y, and z is

$$\begin{vmatrix} 0 & p^\alpha & 2p^\alpha & \mu + h \\ 2 & 0 & 2p^\alpha & \mu + k - \overline{F}_0 \\ 2 & p^\alpha & 2p^\alpha & \mu + \delta - \overline{\Gamma}_0 \\ 1 & p^\alpha - 1 & 2p^\alpha - 1 & \mu - \overline{\Gamma}_1 \end{vmatrix} = 0.$$

This reduces to

$$(p^\alpha - 1)\delta = \mu - h(p^\alpha - 1) - k - (\overline{E}_0 + p^\alpha \overline{\Gamma}_2),$$

whence we conclude that *when the order of st is twice a power of an odd prime, its degree does not exceed $\mu + \mu/(p^\alpha - 1)$.*

5. Let $N = p_1 p_2 \cdots p_r$, the product of r distinct odd prime factors.

There is at once the equation

$$E_1 = \mu,$$

and there are $\sum_{i=1}^{i=r} {}_rC_i = 2^r - 1$ equations given by the $2^r - 1$ powers $(st)^{N/(p_i p_j \cdots)}$, which is in no case the identity but is of degree $\mu + h_{ij}\ldots$, where the quantities $h_{ij}\ldots$ are positive whole numbers or zero. Let $\delta \equiv h_{12\ldots r}$, a constantly recurring quantity since $\mu + \delta$ is the degree of st or of $\{s, t\}$. Now $(st)^{N/p_i p_j}$, say, involves letters from and only from those transitive con-

stituents of $\{s, t\}$ whose degree contains p_i or p_j as a factor, and so on for other combinations of the primes $p_1, p_2, \cdots$. The equation given by $(st)^{N/p_1 p_2}$ when $r = 3$, for example, is

$$p_1 y_1 + p_2 y_2 + p_1 p_2 y_{12} + p_1 p_3 y_{13} + p_2 p_3 y_{23} + p_1 p_2 p_3 y_{123} = \mu + h_{12}.$$

There are $2^r - 1$ variables $y_1, \cdots, y_{12}, \cdots$ in the 2^r equations. Replace these equations by an equivalent system got by replacing the first, $E_1 = \mu$, by $E_0 - E_1 = \delta$ and the last $2^r - 2$ by themselves less the second, $E_0 = \mu + \delta$. The eliminant of these equations is

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \delta \\ p_1 & p_2 & p_3 & \cdots & p_2 p_3 \cdots p_r & p_1 p_2 \cdots p_r & \mu + \delta \\ 0 & p_2 & p_3 & \cdots & p_2 p_3 \cdots p_r & p_1 p_2 \cdots p_r & -h_1 + \delta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Multiply the first row of this determinant by $p_1 p_2 \cdots p_r$ and then divide the first column by p_1, the second by $p_2, \cdots$, the $(r+1)$th by $p_1 p_2, \cdots$. Then we have

$$\begin{vmatrix} p_2 p_3 \cdots p_r & p_1 p_3 \cdots p_r & \cdots & p_2 & p_1 & 1 & p_1 p_2 \cdots p_r \delta \\ 1 & 1 & \cdots & 1 & 1 & 1 & \mu + \delta \\ 0 & 1 & \cdots & 0 & 1 & 0 & -h_1 + \delta \\ 1 & 0 & \cdots & 1 & 0 & 0 & -h_2 + \delta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

For the sake of convenience of reference we may call the row in which h_i, $h_{ij}, \cdots$ occur the h_i-row, the h_{ij}-row, and so forth. Then the law of the formation of the last $2^r - 2$ rows is evident: a term in the h_i-row (except $\delta - h_i$) is unity if the term above it in the first row contains p_i as a factor; a term in the h_{ij}-row is unity if the term in the first row above it contains $p_i p_j$ as a factor, and so on; otherwise it is zero. For wherever a zero is found in the h_{ijk}-row of the determinant there was a term $p_i p_j p_k y_{ijk}$ in the original equation, from which was subtracted $E_0 = \mu + \delta$, and this product $p_i p_j p_k$ we divided out from every term of the column corresponding to y_{ijk}, thus leaving $p_1 p_2 \cdots p_r/(p_i p_j p_k)$ in the first row directly above the zero in question. The present units correspond to zeros in the original equations. For the sake of clearness it may be well to write out the determinant for the case $r = 3$:

$$\begin{vmatrix} p_3p_2 & p_3p_1 & p_2p_1 & p_3 & p_2 & p_1 & 1 & p_1p_2p_3\delta \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mu+\delta \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & \delta-h_1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & \delta-h_2 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & \delta-h_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \delta-h_{12} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \delta-h_{13} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta-h_{23} \end{vmatrix} = 0.$$

Now multiply the second row by -1, the h_i-row by $-(p_i-1)$, the h_{ij}-row by $-(p_i-1)(p_j-1)$, $\cdots$, and add them all to the first row. The sum in a typical column, the $y_{12\cdots k}$-column, is

$$p_1p_2\cdots p_k - \Big[1+\sum_{i=1}^{k}(p_i-1)+\sum_{i=1;\ j=i+1}^{i=k;\ j=k}(p_i-1)(p_j-1)+\cdots$$
$$+(p_1-1)(p_2-1)\cdots(p_k-1)\Big]$$
$$= p_1p_2\cdots p_k-(p_1-1+1)(p_2-1+1)\cdots(p_k-1+1)=0.$$

This gives zero in every term of the first row except one. From the addition of the elements of the last column we have

$$\delta\,[p_1p_2\cdots p_r-1-\sum(p_1-1)-\sum(p_1-1)(p_2-1)-\cdots$$
$$-\sum(p_1-1)(p_2-1)\cdots(p_{r-1}-1)]$$
$$-\mu+\sum h_1(p_1-1)+\sum h_{12}(p_1-1)(p_2-1)+\cdots=0.$$

Hence when the order of st is the product of r distinct odd primes $p_1, p_2, \cdots, p_r$, the degree of $\{s, t\}$ does not exceed

$$\mu+\mu/(p_1-1)(p_2-1)\cdots(p_r-1).$$

6. Let $N=2pq$, where p and q are two distinct odd primes. The following five equations can be written down:

from s and t jointly, $\quad m+\Gamma_1=\mu,$

from st, $\quad 2m+\Gamma_0=\mu+\delta,$

from $(st)^{pq}$, $\quad 2m+F_0=\mu+h,$

from $(st)^{2q}$, $\quad \Gamma_0-qy_2-2qz_2=\mu+i,$

from $(st)^{2p}$, $\quad \Gamma_0-py_1-2pz_1=\mu+j,$

where δ, h, i, and j are positive whole numbers or zero. The elimination of m, y_1, z_1, and z_3 gives

$$\begin{vmatrix} 1 & p-1 & 2p-1 & 2pq-1 & \mu-(q-1)y_2-(pq-1)y_3-(2q-1)z_2 \\ 2 & p & 2p & 2pq & \mu+\delta-qy_2-pqy_3-2qz_2 \\ 2 & 0 & 2p & 2pq & \mu+h-2qz_2 \\ 0 & p & 2p & 2pq & \mu+i-pqy_3 \\ 0 & 0 & 0 & 2pq & \mu+j-qy_2-pqy_3-2qz_2 \end{vmatrix} = 0,$$

which reduces to

$$-\mu+q(p-1)\delta+qh+(p-1)qi+(q-1)j+q(pq-2p+1)y_2 + pq(q-1)y_3+2q(q-1)(p-1)z_2=0;$$

whence $\delta \leqq \mu/q(p-1)$. *If the order of st is twice the product of two distinct odd primes the degree of $\{s, t\}$ does not exceed*

$$\mu+\mu/[q(p-1)].$$

This case is needed in the proof of our theorem only when one of the two odd primes is 3.

7. Now let N have at least two prime power factors the smaller of which is greater than 3. $N=p^\alpha q^\beta \phi$, where p and q are distinct primes and ϕ is unity or a number relatively prime to p and q. We start from

$$\mu+\delta=2\mu-\Gamma_2.$$

The powers $(st)^{N/p}$ and $(st)^{N/q}$ give

$$\Gamma_0(*,0)+\Gamma_0(*,*)=\mu+h,$$
$$\Gamma_0(0,*)+\Gamma_0(*,*)=\mu+i,$$

where h and i are positive whole numbers or zero, and $\Gamma_a(*,0)$ is to be understood as equal to $\sum y(Y-a)+\sum z(Z-a)$, the summation extending however only to those constituents of $\{s, t\}$ whose degrees are divisible by p^α and not by q^β. In $\Gamma_a(0,*)$ the summation extends to the constituents whose degrees are divisible by q^β and not by p^α, while in $\Gamma_a(*,*)$ the summation extends to constituents whose degrees are divisible by both p^α and q^β. Thus

$$\Gamma_a=\Gamma_a(*,*)+\Gamma_a(*,0)+\Gamma_a(0,*)+\Gamma_a(0,0).$$

Then

$$0=-\mu-\frac{h+i}{2}+\frac{\Gamma_0(*,0)+\Gamma_0(0,*)}{2}+\Gamma_0(*,*).$$

Multiply by $(p^\alpha q^\beta - 2)/(p^\alpha q^\beta)$ and add to

$$\mu + \delta = 2\mu - \Gamma_2(*, 0) - \Gamma_2(0, *) - \Gamma_2(*, *) - \Gamma_2(0, 0),$$

with the result (writing π for $p^\alpha q^\beta$)

$$\mu + \delta = \mu + 2\mu/\pi - \frac{\pi - 2}{\pi}\frac{h + i}{2} - \frac{\pi + 2}{2\pi}\Gamma'_{4\pi/(\pi+2)} - \frac{2}{\pi}\Gamma''_\pi - \Gamma_2^{(2)}.$$

Here the new symbol $\Gamma_a^{(k)}$ is introduced $(k = 0, 1, 2, \cdots)$. It is to mean the sum $\Gamma_a(*, 0) + \Gamma_a(0, *)$ when $k = 1$, $\Gamma_a(*, *)$ when $k = 2$, $\Gamma_a(0, 0)$ when $k = 0$. It will presently be used when more than two prime powers are in question. Then $\Gamma_a^{(3)}$, for example, shall be the sum of all the quantities which, like $\Gamma_a(*, *, *, 0, \cdots)$ contain three stars in the parenthesis. Returning to the proof of the last result,

$$\frac{\pi - 2}{2\pi}\Gamma_0(*, 0) - \Gamma_2(*, 0) = \sum\left(\frac{\pi - 2}{2\pi}Y - Y + 2\right)y$$

$$+ \sum\left(\frac{\pi - 2}{2\pi}Z - Z + 2\right)z = -\sum\left(\frac{\pi + 2}{2\pi}Y - 2\right)y$$

$$-\sum\left(\frac{\pi + 2}{2\pi}Z - 2\right)z = -\frac{\pi + 2}{2\pi}\sum\left(Y - \frac{4\pi}{\pi + 2}\right)y$$

$$-\frac{\pi + 2}{2\pi}\sum\left(Z - \frac{4\pi}{\pi + 2}\right)z = -\frac{\pi + 2}{2\pi}\Gamma_{4\pi/(\pi+2)}(*, 0);$$

and similarly

$$\frac{\pi - 2}{\pi}\Gamma_0(*, *) - \Gamma_2(*, *) = -\frac{2}{\pi}\Gamma_\pi(*, *) = -\frac{2}{\pi}\Gamma''_\pi.$$

The summations $\sum$ are to be taken in accordance with the symbol $\Gamma_0(*, 0)$ or $\Gamma_0(*, *)$. Hence in this case $n \leqq \mu + 2\mu/\pi$.

8. If $q^\beta = 3$ and $p^\alpha > 3$, we may proceed as follows:

$$n = 2\mu - \Gamma_2 + \frac{4}{3p^\alpha}\left(-\mu - h + \Gamma_0(*, 0) + \Gamma_0(*, *)\right)$$

$$+ \frac{p^\alpha - 2}{p^\alpha}\left(-\mu - i + \Gamma_0(0, *) + \Gamma_0(*, *)\right),$$

$$n = 2\mu - \Gamma_2 + \frac{\pi - 6}{\pi}\left(-\mu - h + \Gamma_0(*, 0) + \Gamma_0(*, *)\right)$$

$$+ \frac{4}{\pi}\left(-\mu - i + \Gamma_0(0, *) + \Gamma_0(*, *)\right)$$

$$= \mu + \frac{2\mu}{\pi} - \frac{\pi - 6}{\pi}h - \frac{4}{\pi}i - \frac{\pi}{6}\Gamma_{\pi/3}(*, 0) - \frac{\pi - 4}{\pi}\Gamma_{2\pi/(\pi-4)}(0, *)$$

$$- \frac{2}{\pi}\Gamma_\pi(*, *) - \Gamma_2(0, 0).$$

Hence n is not greater than $\mu + 2\mu/\pi$.

9. Next, let N have at least r ($r > 2$) Sylow divisors each of which is greater than 2: $N = \pi\phi$, where $\pi = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, the product of powers of r distinct primes, and where ϕ is relatively prime to π. From the r substitutions $(st)^{N/p_k}$, $k = 1, 2, \cdots, r$, we have r equations which when added together give

$$-r\mu - \sum_{k=1}^{r} h_k + \sum_{k=1}^{r} k\Gamma_0^{(k)} = 0.$$

Multiply this equation by $(\pi - 2)/r\pi$ and add to

$$n = 2\mu - \sum_{k=0}^{r} \Gamma_2^{(k)},$$

whence

$$n = \mu + \frac{2\mu}{\pi} - \sum_{k=0}^{r} \frac{2k + \pi(r-k)}{r\pi} \Gamma^{(k)}_{2r\pi/[2k+(r-k)\pi]} - \frac{\pi - 2}{r\pi} \sum_{k=1}^{r} h_k.$$

If $k = 0$,

$$\frac{2r\pi}{2k + (r-k)\pi} = 2,$$

and if $k = r$,

$$\frac{2r\pi}{2k + (r-k)\pi} = \pi;$$

since $r > 2$, and $\pi - 2 < r\pi/3$, we have for $k = 1$,

$$\frac{2r\pi}{2 + (r-1)\pi} < 3.$$

When $1 < k < r$,

$$\frac{2r\pi}{2k + (r-k)\pi} < \frac{2r}{r-k} < 2(k+1) < 3 \cdot 4 \cdot 5 \cdots \text{ to } k \text{ factors.}$$

It follows at once that the greatest possible value of n in this case is $\mu + 2\mu/\pi$. The results of the last three paragraphs may be stated thus:

If the order of st has r ($r > 1$) Sylow divisors, no one of which is the first power of 2, and whose product we call π, then the degree of $\{s, t\}$ does not exceed $\mu + 2\mu/\pi$.

10. There remains to be investigated the limit of n when N is twice an odd number (π) in which there are at least two prime power factors.

If $\pi = p^a \cdot 3$, we have already before us

$$n = \mu + \frac{2\mu}{\pi} - \frac{4i}{\pi} - \frac{\pi - 6}{\pi} h - \frac{\pi - 4}{\pi} \Gamma_{2\pi/(\pi-4)}(0, *)$$
$$- \frac{6}{\pi} \Gamma_{\pi/3}(*, 0) - 2z - \Gamma_2(0, 0),$$

where z is the number of transitive constituents of degree 2π in $\{s, t\}$. Now $(st)^\pi$ gives

$$2m + F_0 = \mu + j,$$

where j is a positive integer or zero; expanding this,

$$2\pi z + F_0(*, 0) + F_0(0, *) + F_0(0, 0) = \mu + j - 2m.$$

Substitute this value of $2z$ in the expression for n above, and obtain

$$n = \mu + \frac{\mu}{\pi} + \frac{2m}{\pi} - \frac{6}{\pi} E_{\pi/3}(*, 0) - \frac{\pi - 4}{\pi} E_{2\pi/(\pi-4)}(0, *) - E_2(0, 0)$$
$$- \frac{5}{\pi} F_{2\pi/5}(*, 0) - \frac{\pi - 5}{\pi} F_{2\pi/(\pi-5)}(0, *) - \frac{\pi - 1}{\pi} F_{2\pi/(\pi-1)}(0, 0)$$
$$- \frac{\pi - 6}{\pi} h - \frac{4i}{\pi} - \frac{j}{\pi}.$$

Now

$$2m + \Gamma_0 \leqq \mu + \frac{2\mu}{\pi},$$

as has been proved, while $\Gamma_0 \geqq \mu$. Hence

$$2m \leqq \frac{2\mu}{\pi}, \qquad \frac{2m}{\pi} \leqq \frac{2\mu}{\pi^2},$$

and therefore

$$n \leqq \mu + \frac{\mu}{\pi} + \frac{2\mu}{\pi^2}.$$

11. When $N = 2\pi$ is twice an odd number which is the product of two or more prime powers each of which is greater than 3 we may use the formula

$$n = \mu + \frac{2\mu}{\pi} - \frac{\pi - 2}{r\pi} \sum_{k=1}^{r} h_k - \sum_{k=0}^{r-1} \frac{2k + \pi(r - k)}{r\pi} \Gamma^{(k)}_{2r\pi/[2k+(r-k)\pi]} - 2z,$$

where z is the number of transitive constituents in $\{s, t\}$ of degree 2π. From $(st)^\pi$ we have

$$-2\pi z = -\mu - j + 2m + \sum_{k=0}^{r-1} F_0^{(k)}$$

(j a positive integer or zero), which when substituted for $2z$ in the preceding expression gives

$$n = \mu + \frac{\mu}{\pi} + \frac{2m}{\pi} - \sum_{k=0}^{r-1} \frac{2k + \pi(r - k)}{r\pi} E^{(k)}_{2r\pi/[2k+(r-k)\pi]}$$
$$- \sum_{-0}^{r-1} \frac{2k - r + \pi(r - k)}{r\pi} F^{(k)}_{2r\pi/[2k-r+\pi(r-k)]} - \frac{j}{\pi} - \frac{\pi - 2}{r\pi} \sum_{k=1}^{r} h_k.$$

It is necessary to show that

$$\frac{2r\pi}{2k - r + \pi(r-k)} < 2 \cdot 3 \cdot 5 \cdots \text{ to } k+1 \text{ factors.}$$

If $k = 0$, the difference $Z - 2\pi/(\pi - 1)$ is positive since 6 is the least possible value of Z. If $k = 1$, we must have $2r\pi/(2 - r + r\pi - \pi) < 6$, that is $2r\pi - 3r - 3\pi + 6 > 0$, or $2(r-2)(\pi - \frac{3}{2}) + \pi > 0$, which is certainly true. Finally for $1 < k < r$,

$$\begin{aligned}\frac{r\pi}{2k - r + \pi(r-k)} &< \frac{\pi}{\pi - 1}\frac{r}{r-k} \\ &< \left(1 + \frac{1}{\pi - 1}\right)(k+1) \\ &< 2(k+1) < 3^k < 3 \cdot 5 \cdots \text{ to } k \text{ factors.}\end{aligned}$$

Just as in the preceding paragraph we now draw the conclusion that

$$n \leqq \mu + \frac{\mu}{\pi} + \frac{2\mu}{\pi^2}.$$

When the order of st is twice an odd number π divisible by two or more distinct primes the degree of $\{s, t\}$ is not greater than $\mu + \mu/\pi + 2\mu/\pi^2$.

12. This result is significant in the proof of the present theorem when π is the product of two or more distinct odd primes. In this event it gives a limit for n that is lower than the limit found when the order of st is $\Pi_{k=1}^{r} p_k$, for

$$\frac{1}{\Pi_{k=1}^{r} p_k} + \frac{2}{\Pi_{k=1}^{r} p_k^2} < \frac{1}{\Pi_{k=1}^{r}(p_k - 1)}.$$

In fact,

$$\frac{1}{\Pi_{k=1}^{r} p_k} + \frac{2}{\Pi_{k=1}^{r} p_k^2} < \frac{1}{(p_1 - 1)\Pi_{k=2}^{r} p_k}.$$

It has now been proved that if the order of st is divisible by the odd primes $p_1, p_2, \cdots, p_r$ its degree is at most $\mu + \mu/(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)$, and that if its order is 2^α, $\alpha > 1$, its degree does not exceed $\mu + \mu/2^{\alpha-1}$. Then the theorem follows as stated even when $\{s, t\}$ has regular constituents in letters common to both s and t.

The general proposition that *the degree of a group of class μ generated by two non-commutative substitutions of order 2 and degree μ is at most $\frac{3}{2}\mu$*, is covered by this theorem.*

*Bulletin of the American Mathematical Society, vol. 20 (1914), p. 469.

MULTIPLY TRANSITIVE GROUPS

13. Let G be a p times transitive group, p an odd prime. By hypothesis there is in G a substitution s of order 2 and degree μ:

$$s = (a_0\, a_2)(a_3\, a_4)\cdots(a_{p-2}\, a_{p-1})\cdots,$$

and by reason of the fact that G is more than $p-1$ times transitive, there is in G a second substitution s' similar to s:

$$s' = (a_0\, a_3)(a_2\, a_5)(a_4\, a_7)\cdots(a_{p-3}\, -)\cdots(a_{p-1}).$$

If the blank space that follows a_{p-3} in s' is filled by a letter new to s, the product ss' has at least one cycle of p letters. Transform s' by all the g_{p-1} substitutions of the transitive subgroup of G that leaves fixed the letters $a_0, a_2, \cdots a_{p-1}$. Each of these g_{p-1} transforms $s', s'', \cdots$ has in common with s the $p-2$ letters $a_0, a_2, \cdots, a_{p-2}$, and x other letters. It is easy to calculate $\sum x$ for these g_{p-1} substitutions $s', s'', \cdots$. Any letter, not $a_0, a_2, \cdots$, or a_{p-1}, is found as often in the set as any other. There are in the cycle $(a_{p-3}\, -)$ and in the last $\frac{1}{2}\mu - \frac{1}{2}(p-1)$ cycles of the substitutions of the set $s', s'', \cdots (\mu - p + 2)\, g_{p-1}$ such letters. Hence there are

$$\frac{(\mu - p + 2)\, g_{p-1}}{n - p + 1}$$

substitutions among $s', s'', \cdots$ that contain a given letter (not $a_0, a_2, \cdots, a_{p-1}$). Therefore

$$(1)\quad \sum x = (\mu - p + 1)\frac{(\mu - p + 2)\, g_{p-1}}{n - p + 1} = (\mu - p + 1)(\mu - p + 2)\, g_p.$$

But in the set $s', s'', \cdots$ there are exactly $(n-\mu)\, g_p$ substitutions which replace a_{p-3} by a letter new to s, and $(\mu - p + 1)\, g_p$ that replace a_{p-3} by a letter of s. Then

$$(2)\qquad \sum_{(n-\mu)g_p} (p - 2 + x) \geqq \frac{(p-2)\mu}{p-1}(n-\mu)\, g_p,$$

$$(3)\qquad \sum_{(\mu-p+1)g_p} (p - 2 + x) \geqq \tfrac{1}{2}\mu(\mu - p + 1)\, g_p.$$

A word of explanation should perhaps be added. Since all the substitutions $s', s'', \cdots$ fix a_{p-1}, all are non-commutative with s. Moreover, when s' (say) replaces a_{p-3} by a letter new to s, ss' contains a cycle of degree p. It has been proved above that in the latter event s' has at most $\mu/(p-1)$ letters new to s, whence it follows that the number of letters common to s' and s cannot be less than $\mu - \mu/(p-1) = (p-2)\mu/(p-1)$. Then on the addition of (2) and (3),

$$\sum x \geqq \left(\frac{(p-2)\mu}{p-1} - p + 2\right)(n-\mu)\, g_p + \left(\frac{\mu}{2} - p + 2\right)(\mu - p + 1)\, g_p;$$

put for $\sum x$ its value from (1),

$$(\mu - p + 1)(\mu - p + 2) g_p \geqq \frac{p-2}{p-1}(\mu - p + 1)(n - \mu) g_p + \left(\frac{\mu}{2} - p + 2\right)(\mu - p + 1) g_p,$$

or, on division by $(\mu - p + 1) g_p$, which is not zero because $\mu > p$,*

$$\mu - p + 2 \geqq \frac{p-2}{p-1}(n - \mu) + \frac{\mu}{2} - p + 2,$$

$$\frac{\mu}{2} + \frac{p-2}{p-1}\mu \geqq \frac{p-2}{p-1} n,$$

whence finally,

$$\mu \geqq \frac{2p-4}{3p-5} n.$$

If $p = 3$, that is, if G is triply transitive,

$$\mu \geqq \frac{n}{2}.$$

It will be recalled that the holomorph of the elementary abelian group of order 2^α $(\alpha > 1)$ is a triply transitive group of class $2^{\alpha-1}$, in which all the substitutions of degree $2^{\alpha-1}$ are of order 2.

14. If $p = 5$, this limit is

$$\mu \geqq \tfrac{3}{5} n.$$

Whenever μ is not divisible by 12, this may be replaced by

$$\mu \geqq \frac{3n+4}{5}.$$

For those substitutions s', s'', $\cdots$ which replace a_{p-3} by a letter of s cannot have so many as $\mu/2$ letters in common with s unless such a product as ss' is of order 6. The three equations

$$\begin{aligned} m + 2y + 5z &= \mu, \\ 2m \quad\quad + 6z &= \mu, \\ 3y + 6z &= \mu, \end{aligned}$$

give on solution $z = \mu/12$. Hence if μ is not divisible by 12, no product ss', ss'', $\cdots$ is of order 6, and in consequence

$$\sum_{(\mu-4)g_5} (3 + x) \geqq \left(\frac{\mu}{2} + 1\right)(\mu - 4) g_5,$$

* Jordan, *Traité des substitutions*, 1870, p. 64.

whence

$$\mu \geqq \frac{3n+4}{5}.$$

15. If $p = 7$, no product $ss', ss'', \cdots$ is of order less than 8 when s' ($s'', \cdots$) replaces a_4 by a letter of s. Then for the $(\mu - 6)g_7$ substitutions of the set $s', s'', \cdots$ which replace a_4 by a letter of s, we have

$$\sum_{(\mu-6)g_7} (5+x) \geqq \tfrac{3}{4}\mu(\mu-6)g_7,$$

which added to

$$\sum_{(n-\mu)g_7} (5+x) \geqq \tfrac{5}{6}\mu(n-\mu)g_7,$$

gives on reduction

$$\mu \geqq \tfrac{10}{13}n.$$

16. Let it be assumed that G is more than p times transitive, p an odd prime. Then besides

$$s = (a_0\,a_2)(a_3\,a_4)\cdots(a_{p-2}\,a_{p-1})\cdots(a_p)$$

we have

$$s' = (a_0\,a_3)(a_2\,a_5)\cdots(a_{p-3}\,a_p)\cdots(a_{p-1}).$$

The transitive subgroup G_p of degree $n - p$ transforms s' into a set of g_p substitutions $s', s'', \cdots$ such that all the products $ss', ss'', \cdots$ are of order p or a multiple of p. Hence

$$\sum_{g_p} (p-2+x) \geqq \frac{p-2}{p-1}\mu g_p.$$

Any one of the last $\mu - p + 1$ letters of s is found in

$$\frac{(\mu-p+1)g_p}{n-p} = (\mu-p+1)g_{p+1}$$

of the substitutions $s', s'', \cdots$. Then

$$(\mu-p+1)^2 g_{p+1} \geqq \frac{p-2}{p-1}\mu g_p - (p-2)g_p,$$

or

$$(\mu-p+1)^2 g_{p+1} \geqq \frac{p-2}{p-1}(n-p)(\mu-p+1)g_{p+1},$$

whence

$$\mu \geqq \frac{p-2}{p-1}n + \frac{1}{p-1}.$$

When applied to quadruply transitive groups this limit is

$$\mu \geqq \tfrac{1}{2}n + \tfrac{1}{2},$$

and for sextuply transitive groups,

$$\mu \geqq \tfrac{3}{4}n + \tfrac{1}{4}.$$

17. The general case may now be very briefly stated. Let the group G be more than σ times transitive, where σ is the sum of $r\,(>1)$ distinct odd primes, $p_1, p_2, \cdots, p_r$. When the substitution s is given, s', because G_σ is transitive, may be so written that the first r cycles of ss' are of the orders $p_1, p_2, \cdots, p_r$. The substitutions s and s' have $\sum_{k=1}^{k=r}(p_k-2)$ common letters in these first $\frac{1}{2}\sum_{k=1}^{k=r}(p_k-1)$ cycles, and x other letters in common. Summing for all the g_σ substitutions $s', s'', \cdots$

$$\sum x = \Big(\mu - \sum_1^r (p_k-1)\Big)^2 g_{\sigma+1},$$

and

$$\sum_{g_\sigma}\Big(\sum_1^r (p_k-2)+x\Big) \geqq \frac{\prod_{k=1}^{r}(p_k-1)-1}{\prod_{k=1}^{r}(p_k-1)}\mu g_\sigma,$$

whence, putting $\prod$ for $\prod_1^r(p_k-1)$,

$$\Big(\mu - \sum_1^r (p_k-1)\Big)^2 \geqq \Big[\frac{\prod-1}{\prod}\mu - \sum_1^r (p_k-2)\Big]\Big(n-\sum_1^r p_k\Big),$$

whence

$$\mu - \sum_1^r p_k + r \geqq \frac{\prod-1}{\prod}\Big(n-\sum_1^r p_k\Big) + \frac{r(\prod+1)-\sum_1^r p_k}{\prod}\cdot\frac{n-\sum_1^r p_k}{\mu-\sum_1^r (p_k-1)}.$$

Since

$$n - \sum_1^r p_k > \mu - \sum_1^r (p_k-1),$$

$$\mu > \frac{\prod_1^r (p_k-1)-1}{\prod_1^r (p_k-1)}n + \frac{r}{\prod_1^r (p_k-1)}.$$

Here the condition $r>1$ must be taken into account because the coefficient of $(n-\sum_{k=1}^{k=r}p_k)/\big(\mu-\sum_{k=1}^{k=r}(p_k-1)\big)$ is zero when $r=1$. Then if a group is more than 8 or more than 15 times transitive, μ exceeds $\frac{7}{8}n$ or $\frac{47}{48}n$. In connection with this result it might be well to note that a closer study of our group G when 16 times transitive has shown that instead of the limit $\frac{47}{48}n$, the class is actually greater than $\frac{69}{70}n$. Since this appears to be an isolated result depending on the three primes 3, 5, and 7, I omit the proof.

18. If G is exactly $\sigma = p_1 + p_2 + \cdots + p_r$ times transitive, $p_1, p_2, \cdots, p_r$ being distinct odd primes in ascending order of magnitude, $r > 1$, we can prove in like manner that

$$\mu > \frac{\left(\prod_1^r (p_k - 1) - 1\right) n + r - 1}{\prod_1^r (p_k - 1) + p_1 - 2}.$$

19. Let G be doubly transitive.* The substitution s of order 2 and degree $\mu\,(>3)$ is one of w conjugates. Any transposition occurs exactly

$$w\mu/n(n-1)$$

times in this set. The number of possible transpositions in which one letter is displaced by s and the other is fixed by s is $\mu(n-\mu)$, so that if y is the number of substitutions conjugate to s which are not commutative with s,

$$y \geqq \frac{2w\mu(n-\mu)}{n(n-1)}.$$

Since μ letters may be paired in $\mu(\mu-1)/2$ ways, the set has $w\mu(\mu-1)/2$ pairs, and $w\mu(\mu-1)/n(n-1)$ substitutions of the set have a given pair in common. Hence if x is the number of letters any one of the w substitutions has in common with s,

$$\sum_w x = w\mu^2/n, \qquad \sum_w x(x-1) = w\mu^2(\mu-1)^2/n(n-1).$$

It is assumed that μ is less than $n/2$. Neither y nor $w-y$ is zero. Now

$$\sum_w x^2 = \sum_w \left(x - \frac{1}{w}\sum_w x\right)^2 + \frac{1}{w}\left(\sum_w x\right)^2,$$

whence

$$\sum_w \left(x - \frac{\mu^2}{n}\right)^2 = \frac{w\mu^2(n-\mu)^2}{n^2(n-1)}.$$

Since

$$\sum_y x \geqq y\mu/2,$$

$$\sum_y \left(x - \frac{\mu^2}{n}\right)^2 \geqq y\left(\frac{\mu}{2} - \frac{\mu^2}{n}\right)^2,$$

and therefore

$$\sum_{w-y} \left(x - \frac{\mu^2}{n}\right)^2 \leqq \frac{w\mu^2(n-\mu)^2}{n^2(n-1)} - y\left(\frac{\mu}{2} - \frac{\mu^2}{n}\right)^2.$$

* This page is a brief sketch of the reasoning given in full in the Bulletin of the American Mathematical Society, 2d series, vol. 20 (1914), pp. 471 ff., which see, as well as Bochert, l. c.

Now

$$\sum_{w-y}\left(x-\frac{\mu^2}{n}\right)^2 \geqq \frac{1}{w-y}\left[\sum_{w-y}\left(x-\frac{\mu^2}{n}\right)\right]^2 \geqq \frac{1}{w-y}\,\frac{y^2\mu^2}{n^2}\left(\frac{n}{2}-\mu\right)^2,$$

so that

$$\frac{w\mu^2(n-\mu)^2}{n^2(n-1)} - y\left(\frac{\mu}{2}-\frac{\mu^2}{n}\right)^2 \geqq \frac{1}{w-y}\,\frac{y^2\mu^2}{n^2}\left(\frac{n}{2}-\mu\right)^2,$$

whence finally

$$\frac{n(n-\mu)}{2\mu} - \frac{(n-\mu)^2}{n-1} \geqq \left(\frac{n}{2}-\mu\right)^2.$$

On solution this inequality gives

$$\mu > \tfrac{1}{2}n - \tfrac{1}{2}\sqrt{n} - 1.$$

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SOME GEOMETRIC CHARACTERIZATIONS OF ISOTHERMAL NETS ON A CURVED SURFACE*

BY

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INTRODUCTION

If a surface S, whose equations are

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

be referred to an orthogonal net of parameter curves, its first fundamental form is

$$ds^2 = Edu^2 + Gdv^2,$$

where E and G are functions of u and v. This orthogonal net is called *isothermal*, if by a proper transformation $\bar{u} = \phi(u)$, $\bar{v} = \psi(v)$ of the parameters the first fundamental form may be reduced to

$$ds^2 = \overline{\lambda}(\bar{u}, \bar{v})(d\bar{u}^2 + d\bar{v}^2).$$

A necessary and sufficient condition that such a reduction may be effected upon the form as first written is, that the equation

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{E}{G}\right) = 0 \tag{1}$$

be satisfied identically.

So far as the writer is aware, a purely geometric interpretation of this analytic condition does not appear in the literature. Of course the characterization of an isothermal net as a net which divides the surface into infinitesimal squares is not a geometric one.† The fact that these nets have long played a very important part in geometric investigations may lend interest to a purely geometric characterization thereof. In what follows, we present two such characterizations, each of which has its own peculiar

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† Nor can Kasner's elegant characterization of an isothermal net, as one for which the two-parameter family of isogonal trajectories form a linear system, be regarded as geometric. Cf. his note, *A characteristic property of isothermal systems of curves*, Mathematische Annalen, vol. 59 (1904), pp. 352–4.

advantages: one is very easily derived, while the other, although it requires more complicated analytic machinery, leads to a more elegant result.

It should be noted, moreover, that the second characterization mentioned applies to all interesting cases, whereas the first breaks down when the isothermal net consists of the lines of curvature, that is, when the surface is isothermic. Darboux, however, has given a very elegant, though more complicated, geometric characterization of isothermic surfaces,* in connection with his solution of the problem of Ribaucour, in terms of the envelope of a certain two-parameter family of spheres. The related transformations D_m of isothermic surfaces, due to Darboux, permit the characterization of such surfaces in terms of the theory of rectilinear congruences.

1. The relation R. Conjugate congruences

Let X, Y, Z be the direction cosines of a line l in space. If X, Y, Z be given as functions of u, v, then through each point $P(x, y, z)$ of the surface S passes a line l, and the totality of these lines form a congruence which we shall denote by Γ. We shall assume that the line l which corresponds to a point P of S does not lie in the plane tangent to S at P, in other words, that *the determinant*

$$\Delta \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & X \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & Y \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & Z \end{vmatrix}$$

does not vanish for any point of the surface S.

Let us write the equations

$$\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} + cX, \\ \frac{\partial^2 x}{\partial u \partial v} &= a'\frac{\partial x}{\partial u} + b'\frac{\partial x}{\partial v} + c'X, \\ \frac{\partial^2 x}{\partial v^2} &= a''\frac{\partial x}{\partial u} + b''\frac{\partial x}{\partial v} + c''X, \end{aligned} \tag{2}$$

the coefficients of which are functions of u, v which shall be determined as follows. In the first of equations (2), for instance, replace x, X successively by y, Y and by z, Z, thus obtaining three equations in ail, which may be

* *Sur les surfaces isothermiques*, Annales scientifiques de l'école normale supérieure, 3e série, vol. 16 (1899), pp. 491–508, in particular p. 504.

solved for a, b, c since the determinant Δ is nowhere zero. The coefficients of the second and third of equations (2) may be calculated in the same way. These equations are obviously fundamental in the study of the congruence Γ in its relation to the surface S. They are, in fact, very similar to the Gauss equations for the surface S, to which they reduce if X, Y, Z are the direction cosines of the normal to S.

The equations

$$\xi = x + \mu \frac{\partial x}{\partial u}, \qquad \eta = y + \mu \frac{\partial y}{\partial u}, \qquad \zeta = z + \mu \frac{\partial z}{\partial u} \tag{3}$$

define the coördinates of a point $M(\xi, \eta, \zeta)$ on the tangent to the curve $v = \text{const.}$ at P. If P be allowed to move along a curve $u = \text{const.}$ on S, the corresponding lines PM generate a ruled surface $R^{(u)}$, and the point M traces a curve on this ruled surface. The tangent at any point of this curve traced by M has its direction cosines proportional to $\partial\xi/\partial v$, $\partial\eta/\partial v$, $\partial\zeta/\partial v$, where for instance

$$\frac{\partial \xi}{\partial v} = \frac{\partial x}{\partial v} + \frac{\partial \mu}{\partial v}\frac{\partial x}{\partial u} + \mu \frac{\partial^2 x}{\partial u \partial v}.$$

Using the second of equations (2), we obtain

$$\frac{\partial \xi}{\partial v} = \left(\frac{\partial \mu}{\partial v} + \mu a'\right)\frac{\partial x}{\partial u} + (1 + \mu b')\frac{\partial x}{\partial v} + \mu c' X. \tag{4}$$

The tangent to the curve traced by the point M as v alone varies lies in the plane determined by the lines l and PM if and only if

$$\mu = -\frac{1}{b'}. \tag{5}$$

We have therefore determined geometrically a unique point M on each line PM, which we shall call the *minus first transform* of the point P, and which is defined by the equations

$$\xi = x - \frac{1}{b'}\frac{\partial x}{\partial u}, \qquad \eta = y - \frac{1}{b'}\frac{\partial y}{\partial u}, \qquad \zeta = z - \frac{1}{b'}\frac{\partial z}{\partial u}. \tag{6}$$

In the same way, we may determine the *first transform* of the point P as a point M' lying on the tangent to the curve $u = \text{const.}$ at P, and defined by the equations

$$\xi' = x - \frac{1}{a'}\frac{\partial x}{\partial v}, \qquad \eta' = y - \frac{1}{a'}\frac{\partial y}{\partial v}, \qquad \zeta' = z - \frac{1}{a'}\frac{\partial z}{\partial v}. \tag{7}$$

Our designation of the points M and M' as the minus first and first transforms of the point P is intended to indicate the analogy to the Laplace trans-

forms of a conjugate net. In fact, if the parametric net on S is conjugate, $c' = 0$, the second of equations (2) becomes one of the Laplace type, and the points M and M' defined by equations (6) and (7) are actually the minus first and first Laplace transforms of the point P.

If the parametric net is conjugate, the points M and M' just determined are the same no matter what the congruence Γ may be. If, however, the parametric net is not conjugate, the points M and M' change with the congruence Γ. The following considerations will show why this is so. For a non-conjugate parametric net, the ruled surface $R^{(u)}$ formed by the tangents to the curves $v = \text{const.}$ along a fixed curve $u = \text{const.}$ is a skew ruled surface, and therefore a plane through a generator thereof is tangent to the ruled surface at a definite point of that generator. From the way in which the point M was obtained, it is obvious that *the point M is the point at which the plane determined by the lines l and PM is tangent to the ruled surface $R^{(u)}$*. A similar characterization obtains for the point M', and so different points M and M' would be determined by different congruences Γ.

The line l' which joins the points M and M' we shall call the *line conjugate to the line l*, and the congruence Γ' which is formed by the lines l' the *congruence conjugate to the congruence Γ*. If the parametric net is non-conjugate, it is easy to see that a given congruence Γ' can have but one congruence Γ associated with it, so that the congruences Γ and Γ' may in this case be described as conjugate to each other. For, the line l' intersects the parametric tangents in points M and M', and the line l is then determined as the line of intersection of the two planes tangent at M and M' to the two parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$ respectively. If the parametric net is conjugate, however, a congruence Γ will have a unique conjugate congruence Γ', but a congruence Γ'—of which there exists only one for a given conjugate net—will be conjugate to any congruence Γ. In the case of a conjugate net, Wilczynski* has called the line joining the minus first and first Laplace transforms of a point P of the surface the *ray* of the point P, and the congruence formed of these lines the *ray congruence*.

We have elsewhere† spoken of the relation between the congruences Γ and Γ' as the relation R. It is obviously a purely projective relation, and we have in fact found the notion of the greatest importance in the general theory of surfaces and rectilinear congruences.

* E. J. Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), pp. 311–327.

† In three communications to the Society, abstracts of which have appeared in the Bulletin as follows: vol. 21 (1914–15), pp. 484–5, vol. 22 (1915–16), p. 274, and vol. 23 (1916–17), pp. 73–4.

2. The developables of the congruence Γ'

The congruence Γ' formed of the lines MM' contains two one-parameter families of developables, which we now proceed to determine. Any curve on the surface S may be defined by equations of the form $u = u(t)$, $v = v(t)$. We seek those functions $u(t)$ and $v(t)$ which define the curves on S which correspond to the developables of Γ'. We note first of all that if the line MM' generates a developable, then the tangent to the curve traced by M, the tangent to the curve traced by M', and the generator MM' lie in a plane. The tangent to the curve traced by M has its direction cosines proportional to $d\xi/dt$, $d\eta/dt$, $d\zeta/dt$, where ξ, η, ζ have the values given by equations (6) when u and v are replaced by the functions $u(t)$ and $v(t)$ respectively. We have of course

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial u}\frac{du}{dt} + \frac{\partial \xi}{\partial v}\frac{dv}{dt}, \qquad \text{etc.},$$

where

$$(8)\qquad \begin{aligned} \frac{\partial \xi}{\partial u} &= \left[1 - \frac{a}{b'} - \frac{\partial}{\partial u}\left(\frac{1}{b'}\right)\right]\frac{\partial x}{\partial u} - \frac{b}{b'}\frac{\partial x}{\partial v} - \frac{c}{b'}X, \\ \frac{\partial \xi}{\partial v} &= -\frac{1}{b'}\left(\frac{\partial}{\partial v}\log b' + a'\right)\frac{\partial x}{\partial u} - \frac{c'}{b'}X. \end{aligned}$$

Equations (8) are obtained directly from equations (6), use being made of the first two of equations (2). We have therefore

$$\frac{d\xi}{dt} = \left\{\left(1 - \frac{a}{b'} + \frac{1}{b'^2}\frac{\partial b'}{\partial u}\right)\frac{du}{dt} - \frac{1}{b'^2}\left(\frac{\partial b'}{\partial v} + a'b'\right)\frac{dv}{dt}\right\}\frac{\partial x}{\partial u} - \frac{b}{b'}\frac{du}{dt}\frac{\partial x}{\partial v} - \frac{1}{b'}\left(c\frac{du}{dt} + c'\frac{dv}{dt}\right)X,$$

the same equation subsisting, of course, if ξ be replaced successively by η and ζ, and x at the same time by y and z, X by Y and Z. Similarly,

$$\frac{d\xi'}{dt} = -\frac{a''}{a'}\frac{dv}{dt}\frac{\partial x}{\partial u} + \left\{\left(1 - \frac{b''}{a'} + \frac{1}{a'^2}\frac{\partial a'}{\partial v}\right)\frac{dv}{dt} - \frac{1}{a'^2}\left(\frac{\partial a'}{\partial u} + a'b'\right)\frac{du}{dt}\right\}\frac{\partial x}{\partial v} - \frac{1}{a'}\left(c''\frac{dv}{dt} + c'\frac{du}{dt}\right)X.$$

We have also

$$\xi - \xi' = -\frac{1}{b'}\frac{\partial x}{\partial u} + \frac{1}{a'}\frac{\partial x}{\partial v}, \qquad \text{etc.}$$

We wish now to write a necessary and sufficient condition that the line whose direction cosines are proportional to $d\xi/dt$, $d\eta/dt$, $d\zeta/dt$, the line whose direction cosines are proportional to $d\xi'/dt$, $d\eta'/dt$, $d\zeta'/dt$, and the

line whose direction cosines are proportional to $\xi - \xi'$, $\eta - \eta'$, $\zeta - \zeta'$, be parallel to the same plane. The condition sought is easily seen to be the vanishing of the determinant formed from the nine coefficients of $\partial x/\partial u$, $\partial x/\partial v$, X in the above expressions for $d\xi/dt$, $d\xi'/dt$, $\xi - \xi'$. When this determinant is expanded, the condition is found to be

$$A\left(\frac{du}{dt}\right)^2 + 2B\frac{du}{dt}\frac{dv}{dt} + C\left(\frac{dv}{dt}\right)^2 = 0, \tag{9}$$

where

$$A = c'\left[b'(b' - a) + \frac{\partial b'}{\partial u} - a'b\right] + c\left(\frac{\partial a'}{\partial u} + a'b'\right),$$

$$C = -c'\left[a'(a' - b'') + \frac{\partial a'}{\partial v} - a''b'\right] - c''\left(\frac{\partial b'}{\partial v} + a'b'\right),$$

$$B = \frac{1}{2}\left\{c''\left[b'(b' - a) + \frac{\partial b'}{\partial u} - a'b\right] + c'\left(\frac{\partial a'}{\partial u} - \frac{\partial b'}{\partial v}\right) - c\left[a'(a' - b'') + \frac{\partial a'}{\partial v} - a''b'\right]\right\}.$$

Equation (9) is a quadratic differential equation of the first order, the solutions of which determine the curves on the surface S which correspond to the developables of the congruence Γ' of lines MM'. The asymptotic net of the surface S is moreover defined by the quadratic differential equation

$$D\left(\frac{du}{dt}\right)^2 + 2D'\frac{du}{dt}\frac{dv}{dt} + D''\left(\frac{dv}{dt}\right)^2 = 0, \tag{10}$$

where D, D', D'' have the usual significance as the coefficients of the second fundamental form of S. Each of the quadratics (9) and (10) determines a pair of directions at any point of the surface S, and *these pairs separate each other harmonically if and only if*

$$AD'' - 2BD' + CD = 0. \tag{11}$$

The left-hand member of this equation is the simultaneous invariant of the two binary forms (9), (10). But two tangents are conjugate if they separate harmonically the asymptotic tangents. Therefore the net of curves defined by equation (9) is a conjugate net if and only if equation (11) holds.

Now, the coefficients c, c', c'' of equations (2) are easily seen to be proportional to D, D', D''; in fact,

$$c : D = c' : D' = c'' : D'' = \sqrt{EG - F^2}/\Delta,$$

where Δ is the determinant introduced at the beginning of § 1. If the values

of c, c', c'' be substituted in the expressions for A, B, C, and the resulting quantities then substituted in equation (11), this equation reduces essentially to

$$(DD'' - D'^2)\left(\frac{\partial a'}{\partial u} - \frac{\partial b'}{\partial v}\right) = 0. \tag{12}$$

The first factor on the left vanishes identically only for a developable surface. But for such a surface the only conjugate nets are nets one component family of which consists of the straight line generators. We may therefore state the following theorem:

THEOREM. *The developables of the congruence Γ' of lines MM' correspond to a conjugate net on the surface S if and only if*

$$\frac{\partial a'}{\partial u} - \frac{\partial b'}{\partial v} = 0. \tag{13}$$

If the parametric net is conjugate, equation (13) expresses the fact that the net has equal Laplace-Darboux invariants. The characterization of a conjugate net with equal invariants as a net for which the developables of the ray congruence correspond to a conjugate net on the surface, is due to Wilczynski.*

We have found a geometric property which is expressed by equation (13) when the surface S is not developable. We proceed now to obtain another geometric property which covers the case of a developable surface, but which on the other hand becomes trivial if the parametric net on the surface is conjugate.

The totality of points $M(\xi, \eta, \zeta)$ form a surface S_{-1}, which we may call the *minus first transform* of S, and the points $M'(\xi', \eta', \zeta')$ likewise form a surface S_1, the *first transform* of S. From the way in which the point M was obtained, it is clear that the plane of the lines l and PM is tangent to the surface S_{-1} at M. Consequently the tangent at M to the curve $u = \text{const.}$ on S_{-1} intersects the line l in a point whose coördinates we shall now find. The said tangent has direction cosines proportional to $\partial\xi/\partial v$, $\partial\eta/\partial v$, $\partial\zeta/\partial v$, these derivatives being given by the second of equations (8):

$$\frac{\partial \xi}{\partial v} = -\frac{1}{b'}\left(\frac{\partial}{\partial v}\log b' + a'\right)\frac{\partial x}{\partial u} - \frac{c'}{b'}X. \tag{14}$$

The required point of intersection must have for its x-coördinate an expression of the form $\xi + \nu\partial\xi/\partial v$, and also of the form $x + \omega X$. By combining equation (14) with the first of equations (6), it is readily seen that the point of

* Loc. cit., p. 319.

intersection has for its x-coördinate

$$\xi - \frac{1}{\frac{\partial}{\partial v}\log b' + a'}\frac{\partial \xi}{\partial v} = x + \frac{c'}{b'\left(\frac{\partial}{\partial v}\log b' + a'\right)}X. \tag{15}$$

Similarly, the tangent at M' to a curve $v = \text{const.}$ on S_1 meets the corresponding line l in the point whose x-coördinate is

$$\xi' - \frac{1}{\frac{\partial}{\partial u}\log a' + b'}\frac{\partial \xi'}{\partial u} = x + \frac{c'}{a'\left(\frac{\partial}{\partial u}\log a' + b'\right)}X. \tag{16}$$

Examination of the right-hand members of equations (15) and (16) shows that *the points in which the line l is cut by the corresponding tangent to a curve $u = \text{const.}$ on S_{-1} and the corresponding tangent to a curve $v = \text{const.}$ on S_1 coincide if and only if*

$$\frac{\partial a'}{\partial u} = \frac{\partial b'}{\partial v}. \tag{13}$$

If the parametric net on S is conjugate, $c' = 0$, and condition (13) is not at all necessary. In fact, in this case the two points of intersection on l coincide with the point P, always.

3. Geometric characterization of isothermal nets

We shall now suppose that the parametric net on the surface S is an orthogonal net, and that the line l is normal to S at P. Then equations (2) become the Gauss equations of the surface, and in addition $F = 0$. Some of the coefficients in the Gauss equations are

$$c = D, \qquad c' = D', \qquad c'' = D'',$$

$$a' = \frac{1}{2}\frac{\partial}{\partial v}\log E, \qquad b' = \frac{1}{2}\frac{\partial}{\partial u}\log G.$$

The condition that the parametric net be isothermal, i. e., equation (1), may therefore be written

$$\frac{\partial a'}{\partial u} = \frac{\partial b'}{\partial v},$$

which is precisely the equation for which we have already found geometric interpretations. Using the results of the preceding paragraphs, we may therefore state the following geometric characterizations of isothermal nets:

An orthogonal net on a non-developable surface is an isothermal net, if and only if the developables of the congruence which is conjugate to the congruence of normals of the surface correspond to a conjugate net on the surface.

If an orthogonal net on any surface S is not conjugate—i. e., does not consist of the lines of curvature of S—then the net is isothermal if and only if each normal to the surface is cut in the same point by the corresponding tangent to the curve $u = const.$ on the minus first transform of S, and by the corresponding tangent to the curve $v = const.$ on the first transform of S.

If the lines of curvature form an isothermal net, the surface is isothermic; therefore, in virtue of the theorem of Wilczynski, *a surface is isothermic if and only if the developables of the ray congruence of its lines of curvature correspond to a conjugate net on the surface.*

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A NEW METHOD IN BOUNDARY PROBLEMS FOR DIFFERENTIAL EQUATIONS*

BY

R. G. D. RICHARDSON

INTRODUCTION

One of the most important problems of mathematical physics is the study of boundary problems for differential equations of the second order in one or more variables. These equations contain a parameter which must be so determined that solutions exist satisfying prescribed linear homogeneous relations between the values of the function and its derivative on the boundary. The problem in one dimension has been studied in great detail.† As typical of the problems in one independent variable we have the equation

$$(1) \qquad v_{xx} + [r(x) + \lambda k(x)]v(x) = 0, \qquad v(0) = v(1) = 0.$$

By means of his theory of integral equations, Hilbert‡ extended the theory of boundary problems in one dimension to the corresponding problem in partial differential equations of the elliptic type, of which

$$(2) \quad u_{xx} + u_{yy} + (r(x, y) + \lambda k(x, y))u = 0, \qquad u = 0 \text{ on boundary,}$$

is typical. By hypothesis, for both equations (1) and (2) the function k has one sign (orthogonal case) or k has both signs and $r \leqq 0$ (polar case). It has been shown that there exist an infinite set of real parameter values λ_i for

* Presented to the Society, September, 1914.

† For existing methods and literature of the subject see Bôcher, *Encyklopaedie der mathematischen Wissenschaften*, IIA7a; *Boundary problems in one dimension*, Proceedings of the International Congress of Mathematicians (Cambridge, 1912), and *Leçons sur les méthodes de Sturm* (Paris, 1917); Lichtenstein, *Zur Analysis der unendlichvielen Variabeln*, Rendiconti del Circolo Matematico di Palermo, vol. 38, p. 113.

‡ *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, p. 58 (Teubner, 1912). This book is a reprint of a series of articles in the Göttinger Nachrichten.

Refinements concerning the nature of the field R, of the coefficients of the equation and of the functions $f(x, y)$ to be expanded in terms of the set of solutions constitute the chief progress made in this field since Hilbert published his work. See the literature cited by Lichtenstein, *Randwertaufgaben der Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Journal für Mathematik, vol. 142, (1912) p. 1.

which solutions of (1) and (2) exist satisfying prescribed boundary conditions. In the orthogonal case this set has a limiting point at positive infinity only and in the polar at both positive and negative infinity.

The boundary problem for partial differential equations of the hyperbolic type so far discussed has been of an entirely different nature. The existence of a unique solution has been shown* in case the value of the function is given on one of two intersecting curves and the value of the normal derivative on the other, neither curve being cut twice by a characteristic. The elliptic and hyperbolic types of equations are essentially different and are characterized by Du Bois Reymond as "himmelweit verschieden." On the other hand, in certain special cases at least, a boundary problem for the latter, analogous to that for the elliptic case can be solved, and functions $f(x, y)$ may be developed in terms of the solutions corresponding to the various characteristic numbers. A simple example is given by the solutions of the equations

[3(a)] $$u_{xx} + u_{yy} + \lambda u = 0;$$

[3(b)] $$u_{xx} - u_{yy} + \lambda u = 0.$$

If we demand that solutions of these equations shall vanish on the boundary of a square whose opposite vertices are $(0, 0)$, (π, π) we find that solutions $\sin mx \sin ny$ (m, n positive integers) of one equation are solutions of the other. For the elliptic equation the characteristic numbers must be of the form $m^2 + n^2$, for the hyperbolic, $m^2 - n^2$. In the latter case the set has limiting points at 0 and at $\pm \infty$ while in the former this occurs at $+\infty$ only. If the field is taken with sides of incommensurable length there may be an infinite number of limiting points for the set of characteristic numbers of [3(b)].†

Including [3(a)] and [3(b)] as special cases there is a class of partial differential equations for which known methods for ordinary equations suffice. For, if in the unit square we consider the equation

$$(4) \quad m(y)\frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] - s(x)\frac{\partial}{\partial y}\left[q(y)\frac{\partial u}{\partial y}\right] + [r(x)m(y) - s(x)l(y)]u + \lambda s(x)m(y)u = 0$$

for which solutions are to vanish on the boundary, we know that the functions $u(x, y) = v(x)w(y)$ and the corresponding parameters $\lambda_{n_1 n_2} = \mu_{n_1} - \nu_{n_2}$

* For the literature of the subject and the most general theorem see Mason, *Selected Topics in the Theory of Boundary Value Problems of Differential Equations*, The New Haven Mathematical Colloquium (Yale University Press, 1910).

† For example, if the opposite vertices are $(0, 0)$, $(\pi, \pi/\sqrt{2})$, the characteristic numbers are of the form $m^2 - 2n^2$. Among the integers which occur among this set, each an infinite number of times, are all those of the form $H^2 - 2$, where H is an integer.

will fulfil the conditions, $v(x)$ and $w(y)$ being solutions of the equations each in one variable

$$\frac{d}{dx}\left[p(x)\frac{dv(x)}{dx}\right] + [r(x) + \mu s(x)]v(x) = 0,$$

$$p > 0,\ s > 0,\ v(0) = v(1) = 0,$$

$$\frac{d}{dy}\left[q(y)\frac{dw(y)}{dy}\right] + [l(y) + \nu m(y)]w(y) = 0,$$

$$q > 0 \text{ or } q < 0,\ m > 0,\ w(0) = w(1) = 0$$

and μ_{n_1}, ν_{n_2} the parameter values corresponding to solutions which vanish $n_1 - 1$, $n_2 - 1$ times respectively. This is immediately evident on multiplying the first equation by $m(y)w(y)$ and the second by $-s(x)v(x)$ and adding. If $q > 0$ the equation is hyperbolic; if $q < 0$, elliptic.

None of the methods at present in use lends itself to the discussion of the new boundary problem of the hyperbolic equation. *The objects of this memoir are* (*a*) *to develop a new method by which the well-known results in regard to boundary problems for the ordinary and the elliptic partial equations can be deduced and* (*b*) *to attempt an application of this method to an investigation of the facts concerning the new boundary problem for the hyperbolic equation.*

The new method consists in dividing the region into equal rectangles by a lattice and studying the difference equations approximating to the differential equation at the lattice points. The size of the lattice division is decreased indefinitely and it is shown (§ 3) in all cases (including the hyperbolic) that each set of corresponding parameter values λ has a finite limit. The set of numbers which are limits of the characteristic values of the difference has, both in the elliptic and hyperbolic cases, a limit point at infinity. *In the case of the elliptic partial differential equation it is shown* (§ 5) *that the corresponding functions of the lattice approach uniformly a function* $u(x, y)$ *which is continuous and that this limiting function has continuous derivatives of as high order as those possessed by the coefficients of the equation.* The totality of such functions furnishes the characteristic solutions of the equation.

The results obtained in the hyperbolic case (§ 6) suggest that with the possible exception of special cases, the boundary problem similar to that for the elliptic case may be possible of solution. However, the reasoning is not complete and the problem is left in large measure unsolved.

Although the problem in one dimension is not treated by the new method, its discussion is naturally simpler than that of the elliptic equation which is discussed in full. *All the ordinary theorems both for the orthogonal and polar cases of* (1) *can be derived in simple fashion by the new processes.* The method can also be extended (*a*) to problems with more general homogeneous boundary conditions and (*b*) to problems in three or more dimensions.

Beside a completion of the discussion of the hyperbolic case, other new

and interesting problems of various kinds are suggested by the results obtained. No problem in quadratic forms or in linear equations in an infinite number of variables in which the set of characteristic numbers has limit points both at infinity and in the finite region has as yet been solved. We can however set up by a method analogous to that used by Lichtenstein* for ordinary differential equations, problems of this nature equivalent to the hyperbolic case of (4) for which we know solutions exist. The hyperbolic differential equation may be regarded as the Lagrange equation of an isoperimetric problem in the calculus of variations in which there are an infinite number of integral sub-conditions (see footnote, p. 496). Such a problem has never been studied. The expansion of linear forms in terms of the solutions of the approximation equations (§ 1, III–IV) suggests an expansion of functions in terms of the characteristic functions of our equation. The author hopes later to report progress along some of these lines.

1. The linear algebraic equations of approximation

Let us approximate by algebraic equations to the most general linear self-adjoint homogeneous differential equation of the second order in two independent variables which may be written

$$\frac{\partial}{\partial x}\left[p(x,y)\frac{\partial u(x,y)}{\partial x}\right]-\frac{\partial}{\partial y}\left[q(x,y)\frac{\partial u(x,y)}{\partial y}\right]+[r(x,y)+\lambda k(x,y)]u(x,y)=0, \tag{5}$$

where λ is a parameter. The notation will be simplest if the region to be considered is a unit square with opposite vertices $(0,0)$, $(1,1)$ and there are $m\mu$ equal rectangles formed by $m-1$ lines parallel to the y-axis and $\mu-1$ lines parallel to the x-axis, the values of the functions p, q, r, k at one of the lattice points with coördinates $(i/m, j/\mu)$ $(i=0,1,\cdots,m;\ j=0,1,\cdots,\mu)$ being denoted by $p_{ij}, q_{ij}, r_{ij}, k_{ij}$, and the derivatives u_x, u_y being replaced by $m(u_{ij}-u_{i-1j})$, $\mu(u_{ij}-u_{ij-1})$ respectively. In this way we avoid the complexity of notation that arises in the general region for lattice points near the boundary.† *For the algebraic equations it will be assumed*

* Loc. cit.

† More generally the plane may be divided into rectangles by lines parallel to the axes, the lattice points denoted by (i,j), the derivatives $\partial u/\partial x$ at these points replaced by $(u_{ij}-u_{i-1j})/l_i$ where l_i is the length of the interval between $i-1, j$ and i, j, the derivative $\partial^2 u/\partial x^2$ replaced by

$$\frac{2}{l_{i+1}+l_i}\left(\frac{u_{i+1j}-u_{ij}}{l_{i+1}}-\frac{u_{ij}-u_{i-1j}}{l_i}\right),$$

etc. For points near the boundary the lengths of the intervals cannot be made uniform and the equations set up for these points must be of irregular form. While in the following discussion details are given only for the case that the region is a square it should be borne in mind that this restriction is due to a desire to retain simplicity in the notation and that the results hold for more general regions.

that p, q, r, and k are continuous functions of x and y and* (to insure the parameter values being real) *that k is positive throughout the region. For boundary conditions we postulate† that u_{ij} vanish on the contour,*

$$u_{0j} = u_{mj} = u_{i0} = u_{i\mu} = 0 \qquad (i = 0, 1, 2, \cdots, m;\ j = 0, 1, 2, \cdots, \mu). \tag{6}$$

There are then $(m-1)(\mu-1)$ homogeneous equations in as many unknowns of the type

$$\begin{aligned} L(u_{ij}) + \lambda k_{ij} u_{ij} \equiv\ & m^2 [\, p_{i+1j}(u_{i+1j} - u_{ij}) - p_{ij}(u_{ij} - u_{i-1j})\,] \\ & - \mu^2 [\, q_{ij+1}(u_{ij+1} - u_{ij}) - q_{ij}(u_{ij} - u_{ij-1})\,] \\ & + (r_{ij} + \lambda k_{ij}) u_{ij} = 0 \\ & (i = 1, \cdots, m-1;\ j = 1, \cdots, \mu - 1). \end{aligned} \tag{A}$$

If we hold fixed the lattice point in question and indefinitely increase the number of divisions (in this paper we shall consider in general the set obtained by repeatedly doubling m and μ) the limit of (A) gives the equation (5) for this point.

Intimately connected with equations (A) are quadratic forms

$$\begin{aligned} Q_1 \equiv\ & \frac{1}{m\mu} [\, m^2 \sum_{ij} p_{i+1j}(u_{i+1j} - u_{ij})^2 \\ & - \mu^2 \sum_{ij} q_{ij+1}(u_{ij+1} - u_{ij})^2 - \sum_{ij} r_{ij} u_{ij}^2 \,], \\ Q_2 \equiv\ & \frac{1}{m\mu} \sum_{ij} k_{ij} u_{ij}^2, \end{aligned} \tag{7}$$

in variables u_{ij}. From the hypothesis on $k(x, y)$ it follows that Q_2 is definite. In seeking the minimum of Q_1 subject to the restrictions (6) and $Q_2 = 1$, one is led on differentiation with regard to each of the variables u_{ij} to the system of equations (A). Another relation under the conditions (6) is given by the formula

$$Q_1 = -\frac{1}{m\mu} \sum_{ij} L(u_{ij}) u_{ij} = \lambda Q_2, \tag{8}$$

which is established by multiplying the equations (A) by u_{ij}, adding, and dividing by $m\mu$.

In order that there be solutions different from zero of the $(m-1)(\mu-1)$ linear equations (A), λ must be a root of the determinant formed from the

* Eventually the important cases are those where p and q have each one sign. When they have opposite signs ($p > 0$, $q < 0$) we shall call the difference equations elliptic and when the same sign ($p > 0$, $q > 0$) hyperbolic.

† More generally it might be assumed that on the boundary $h_{ij} u_{ij} + l_{ij} \Delta_n (u_{ij}) = 0$ where $\Delta_n (u_{ij})$ is a first difference expression corresponding to the normal derivative, and h_{ij}, l_{ij} are sets of constants. The discussion in that case would be similar.

coefficients of that set of equations. This symmetric determinant has $(m-1)(\mu-1)$ roots for λ, and from our hypothesis it follows that these are all real.* For, since λk_{ij} occurs only as part of each term in the main diagonal, we may multiply the row and column in which it stands by $1/\sqrt{k_{ij}}$ without altering the symmetry of the determinant and thus reduce it to the form of the secular equation.† Corresponding to each of these $(m-1)(\mu-1)$ values of λ, called characteristic values, there is a solution u_{ij} of the linear equations (A). These solutions shall be said to be *normalized* when the u's are multiplied by the constant which makes $Q_2 = 1$. The normalized solution corresponding to the characteristic $\lambda^{(\kappa)}$ and which consists of a set of constants each defined for a point $(i/m, j/\mu)$ of the lattice, shall be called a characteristic function and denoted by $U_{ij}^{(\kappa)}$. These sets of solutions $U_{ij}^{(\kappa)}$ and the values $\lambda^{(\kappa)}$ $[\kappa = 1, \cdots, (m-1)(\mu-1)]$ have various properties, some of which we shall now develop.

I. *Characteristic functions $U_{ij}^{(\kappa)}$, $U_{ij}^{(\nu)}$ corresponding to two values of λ are orthogonal,* that is

$$\sum_{ij} k_{ij} U_{ij}^{(\kappa)} U_{ij}^{(\nu)} = 0. \tag{9}$$

To prove this consider a set of equations (A') similar to the set (A) except that $p_{ij}, q_{ij}, r_{ij}, k_{ij}, u_{ij}, \lambda$ are replaced by $p'_{ij}, q'_{ij}, r'_{ij}, k'_{ij}, u'_{ij}, \lambda'$ respectively. Now multiply the equation for u_{ij} in (A) by u'_{ij} those for u'_{ij} in (A') by $-u_{ij}$, add and sum over ij. There results under condition (6) the fundamental relation

$$\begin{aligned}\frac{1}{m\mu}\sum_{ij}(r_{ij}+\lambda k_{ij}-r'_{ij}-\lambda' k'_{ij})\,u_{ij}\,u'_{ij}&\\ =\frac{m}{\mu}\sum_{ij}(p_{ij}-p'_{ij})(u_{ij}-u_{i-1j})(u'_{ij}-u'_{i-1j})&\\ -\frac{\mu}{m}\sum_{ij}(q_{ij}-q'_{ij})(u_{ij}-u_{ij-1})(u'_{ij}-u'_{ij-1}).&\end{aligned} \tag{10}$$

On setting $k_{ij} = k'_{ij}$, $r_{ij} = r'_{ij}$, $p_{ij} = p'_{ij}$, $q_{ij} = q'_{ij}$ we have for two solutions $U_{ij}^{(\kappa)}$, $U_{ij}^{(\nu)}$ of (A) the relation

$$(\lambda^{(\kappa)} - \lambda^{(\nu)}) \sum_{ij} k_{ij} U_{ij}^{(\kappa)} U_{ij}^{(\nu)} = 0$$

* In the problem corresponding to the elliptic differential equation where Q_1 is definite $(p > 0, q < 0, r < 0)$, the roots λ are of course real even if the k's have both signs. But if neither of the two quadratic forms Q_1, Q_2 is definite, at least a part of the λ's and of the solutions will in general be complex. Similarly if characteristic functions exist for the hyperbolic case $(p > 0, q > 0)$ when k takes on both signs, at least a part of the characteristic numbers will in general be complex. The minimum and calculus of variations problems corresponding will then have no meaning.

† Scott and Matthews, *Theory of Determinants* (2d ed.), p. 147.

and when $\lambda^{(\kappa)} \neq \lambda^{(\nu)}$ this gives the desired orthogonal property. If λ is multiple valued ($\lambda^{(\kappa)} = \lambda^{(\nu)}$), the solutions $U_{ij}^{(\kappa)}$, $U_{ij}^{(\nu)}$ may by the usual methods be so chosen that they are orthogonal to one another as well as to the others. The set of $(m-1)(\mu-1)$ solutions thus chosen as orthogonal and normalized is called the *fundamental set.*

II. The fundamental set $U_{ij}^{(\kappa)}$ $[\kappa = 1, 2, \cdots, (m-1)(\mu-1)]$ comprise $(m-1)(\mu-1)$ *linearly independent* sets of $(m-1)(\mu-1)$ constants. For, if there exist constants c_κ such that $\sum_\kappa c_\kappa U_{ij}^{(\kappa)} = 0$ for all ij, on multiplication of each fundamental set by $k_{ij} U_{ij}^{(\nu)}$ and summing over ij and then over κ we get by using the orthogonal properties the equation $c_p = 0$. This equation holds for all values $p = 1, 2, \cdots, (m-1)(\mu-1)$.

III. Any set of $(m-1)(\mu-1)$ constants ϕ_{ij} defined for the lattice points is expressible in terms of the sets of constants $U_{ij}^{(\kappa)}$ as follows:

$$\phi_{ij} = \sum_\kappa c_\kappa U_{ij}^{(\kappa)} \qquad \left(c_\kappa = \frac{1}{m\mu} \sum_{ij} \phi_{ij} k_{ij} U_{ij}^{(\kappa)}\right).$$

For, denoting by R_{ij} the remainder $\phi_{ij} - \sum_\kappa c_\kappa U_{ij}$ and multiplying by $k_{ij} U_{ij}^{(\kappa)}$ and summing, we get $\sum_{ij} k_{ij} R_{ij} U_{ij}^{(\kappa)} = 0$ for all κ. Since the $(m-1)(\mu-1)$ sets of constants $U_{ij}^{(\kappa)}$ are linearly independent by II, the $(m-1)(\mu-1)$ multipliers $k_{ij} R_{ij}$ must all be zero. And since by hypothesis $k_{ij} > 0$, it follows that R_{ij} is zero.

IV. The arbitrary set of constants ϕ_{ij} is not generally expressible linearly in terms of ν of the U_{ij}'s $[\nu < (m-1)(\mu-1)]$ which we denote by $U_{ij}^{(\bar\kappa)}$. However, if we define the coefficients $c_{\bar\kappa}$ to be

$$c_{\bar\kappa} = \frac{1}{m\mu} \sum_{ij} \phi_{ij} k_{ij} U_{ij}^{(\kappa)},$$

the best approximation to ϕ_{ij} is given by the formula $\sum_{\bar\kappa} c_{\bar\kappa} U_{ij}^{(\bar\kappa)}$, if by best approximation is understood that which furnishes $\sum_{ij} k_{ij} [\phi_{ij} - \sum_{\bar\kappa} c_{\bar\kappa} U_{ij}^{(\bar\kappa)}]^2$ with a minimum.

For, in order to obtain a minimum for this last expression we equate to zero the derivatives with regard to $c_{\bar\kappa}$ and obtain the relations

$$\sum_{ij} k_{ij} (\phi_{ij} - \sum_{\bar\kappa} c_{\bar\kappa} U_{ij}^{(\kappa)}) U_{ij}^{(\bar\kappa)} = 0$$

for all $\bar\kappa$. From the orthogonal and normal properties it follows that this may be written

$$c_{\bar\kappa} = \frac{1}{m\mu} \sum_{ij} \phi_{ij} k_{ij} U_{ij}^{(\bar\kappa)}$$

and on taking the second derivatives these relations are found to be sufficient to establish the theorem.

It follows from III that $\phi_{ij} - \sum_{\bar\kappa} c_{\bar\kappa} U_{ij}^{(\bar\kappa)} = \sum_{\bar\kappa} c_{\bar\kappa} U_{ij}^{(\bar\kappa)}$ where κ ranges over

the remaining $(m-1)(\mu-1)-\nu$ of the κ's and $c_{\bar{\kappa}}$ is defined in a manner analogous to $c_{\bar{\kappa}}$. Hence

$$\frac{1}{m\mu}\sum_{ij} k_{ij}\,[\,\phi_{ij} - \sum_{\bar{\kappa}} c_{\bar{\kappa}}\, U_{ij}^{(\bar{\kappa})}]^2 = \sum_{\bar{\kappa}} c_{\bar{\kappa}}^2.$$

And in particular if $\bar{\kappa}$ is a null set,

$$\frac{1}{m\mu}\sum_{ij} k_{ij}\,\phi_{ij}^2 = \sum_{\kappa} c_\kappa^2.$$

V. *Any characteristic number can be considered both as the minimum and the maximum for certain problems. If the* $(m-1)(\mu-1)$ *numbers are arranged in a system* $\lambda_1 \leqq \lambda_2 \leqq \cdots \lambda_{s-1} \leqq \lambda_s \leqq \lambda_{s+1} \leqq \cdots \leqq \lambda_{(m-1)(\mu-1)}$, *then* λ_s *is the minimum* and maximum respectively for those values of* u_{ij} *which make* $Q_2 = 1$ *and satisfy the relation* (6) *and the relations*

[11(*a*)] $$\sum_{ij} k_{ij}\, U_{ij}^{(\kappa)}\, u_{ij} = 0 \qquad (\kappa = 1, \cdots, s-1);$$

[11(*b*)] $$\sum_{ij} k_{ij}\, U_{ij}^{(\kappa)}\, u_{ij} = 0 \quad (\kappa = s+1, \cdots, (m-1)(\mu-1))$$

respectively.

Since proofs for all cases are similar, we shall content ourselves with a discussion of the last case. From III and relation [11(*b*)] it follows that

(12) $$u_{ij} = \sum_{\kappa=1}^{s} c_\kappa\, U_{ij}^{(\kappa)} \qquad \left(c_\kappa = \frac{1}{m\mu}\sum_{ij} k_{ij}\, u_{ij}\, U_{ij}^{(\kappa)}\right)$$

and hence that $Q_2 = c_1^2 + c_2^2 + \cdots + c_s^2 = 1$. From (12) and (*A*) we have

$$L^{(\kappa)}(u_{ij}) = \sum_{\kappa=1}^{s} \lambda_\kappa\, c_\kappa\, k_{ij}\, U_{ij}^{(\kappa)}$$

and hence from (12) and (9)

$$Q_1 = -\sum_{ij} L(u_{ij})\, u_{ij} = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_s c_s^2.$$

Then $Q_1 - \lambda_s = (\lambda_1 - \lambda_s)c_1^2 + \cdots + (\lambda_{s-1} - \lambda_s)c_{s-1}^2$ which is zero or nega-

* As an extension of this result it seems probable that any characteristic function $u(x, y)$ of the hyperbolic differential equation (5) $(p > 0, q > 0)$ may be regarded as the solution of the problem of finding a minimum and maximum respectively of the integral $\int_R (pu_x^2 - qu_y^2 - ru^2)\,dR$ for those values of u which vanish on the boundary, satisfy $\int_R ku^2\,dR = 1$ and an infinite number of orthogonal sub-conditions of the form $\int_R kU_i u\,dR = 0$, where i ranges over the whole set of solutions corresponding to values of λ less and greater respectively than the one considered. The minimum or maximum value would be the corresponding characteristic number λ. The minimum property can be easily established for the elliptic case $(p > 0, q < 0)$, since here the number of sub-conditions is finite; it is probable that the maximum property holds also.

tive, and zero when $c_1 = c_2 = \cdots = c_{s-1} = 0$. That is, Q_1 is a maximum for $U_{ij}^{(s)}$ and takes on value λ_s in this case.

VI. The solution $U_{ij}^{(s)}$ of V furnishes the maximum and minimum $1/\lambda_s$ respectively for Q_2 when those sets of constants u_{ij} are considered which satisfy the relation $Q_1 = 1$, the relations (6) and the relations [11(*a*)], [11(*b*)] respectively.

2. Variation properties of the characteristic numbers of the algebraic equations

The functions $p(x, y)$, $q(x, y)$, $r(x, y)$, $k(x, y)$ may be regarded as analytic functions of a parameter t in an interval $-1 \leqq t \leqq 1$. If in the discussion of I, § 1 we allow p', q', r', k' to approach p, q, r, k, the parameter λ' approaches λ and the formula (10) becomes

$$\frac{d\lambda}{dt}\frac{1}{m\mu}\sum_{ij} k_{ij}u_{ij}^2 = -\frac{1}{m\mu}\sum_{ij}\frac{dr_{ij}}{dt}u_{ij}^2 - \frac{\lambda}{m\mu}\sum_{ij}\frac{dk_{ij}}{dt}u_{ij}^2 + \frac{1}{m\mu}\sum_{ij}\frac{dp_{ij}}{dt}m^2(u_{ij}-u_{i-1j})^2 - \frac{1}{m\mu}\sum_{ij}\frac{dq_{ij}}{dt}\mu^2(u_{ij}-u_{ij-1})^2. \tag{13}$$

From this formula one can determine the method of variation of the parameter λ with the variation of one or more of the functions p, q, r, and k.

I. In the set of equations (*A*) *all of the characteristic values* λ *are increased if one or more of the* p*'s are increased or one or more of the* q*'s decreased* and all are decreased if one or more of the p's are decreased or one or more of the q's increased.

II. In the set of equations (*A*) *all of the values* $|\lambda|$ *are decreased if one or more of the* k*'s are increased.* If $\lambda = 0$, the new characteristic value is also zero.

III. In the set of equations (*A*) *all of the values* λ *are decreased if one or more of the* r*'s are increased.*

It may be observed that since

$$\frac{1}{m\mu}\sum_{ij} k_{ij}u_{ij}^2 \equiv Q_2 = 1,$$

the change of λ is an infinitesimal of the same order as the mean change of the r's and k's. And since neither

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2 \quad \text{nor} \quad \frac{1}{m\mu}\sum_{ij}\mu^2(u_{ij}-u_{ij-1})^2$$

can be zero when $Q_2 = 1$, the change of λ is of at least as high order as the mean changes of the p's or q's.

3. Finiteness of characteristic numbers

The comparison theorems of the preceding section may be used to determine intervals within which the values of λ for all m, μ must lie. In the unit square R with opposite vertices $(0,0)$, $(1,1)$, constants P_1, P_2, Q_1, Q_2, R_1, R_2, may be chosen so that

$$P_1 \geqq \frac{p}{k} \geqq P_2, \qquad Q_1 \leqq \frac{q}{k} \leqq Q_2, \qquad R_1 \leqq \frac{r}{k} \leqq R_2.$$

From I–III of § 2 it follows that the characteristic values of (A) are intermediate between those of

$$m^2 P_1(u_{i+1j} - 2u_{ij} + u_{i-1j}) - \mu^2 Q_1(u_{ij+1} - 2u_{ij} + u_{ij-1}) + (R_1 + \lambda)u_{ij} = 0, \tag{14}$$

$$m^2 P_2(u_{i+1j} - 2u_{ij} + u_{i-1j}) - \mu^2 Q_2(u_{ij+1} - 2u_{ij} + u_{ij-1}) + (R_2 + \lambda)u_{ij} = 0. \tag{15}$$

That the characteristic numbers of (14) and (15) are the differences between the characteristic numbers for the sets of equations

$$\begin{aligned} m^2 P_1(v_{i+1} - 2v_i + v_{i-1}) + (R_1 + \nu_1)v_i &= 0, \\ \mu^2 Q_1(w_{j+1} - 2w_j + w_{j-1}) + \nu_1' w_j &= 0; \end{aligned} \tag{16}$$

$$\begin{aligned} m^2 P_2(v_{i+1} - 2v_i + v_{i-1}) + (R_2 + \nu_2)v_i &= 0, \\ \mu^2 Q_2(w_{j+1} - 2w_j + w_{j-1}) + \nu_2' w_j &= 0, \end{aligned} \tag{17}$$

respectively, where v_i, w_j are each sets of variables in one dimension and ν_1, ν_1', ν_2, ν_2' are parameters, may be seen on multiplying the first of each set by w_j and the other by $-v_i$, adding and setting $u_{ij} = v_i w_j$. The λ of (14) is $\nu_1 - \nu_1'$ and of (15) is $\nu_2 - \nu_2'$.

It is easily shown that characteristic numbers of (16) are approximations for the corresponding differential equations

$$P_1 v_{xx}(x) + (R_1 + \nu_1)v(x) = 0, \qquad v(0) = v(1) = 0, \tag{18}$$

$$Q_1 w_{yy}(y) + \nu_1' w(y) = 0, \qquad w(0) = w(1) = 0, \tag{19}$$

each in one variable and with constant coefficients. These have solutions of the form $v = \sin n_1 \pi x$, $w = \sin n_2 \pi y$ $(n_1, n_2 = 1, 2, \cdots)$, which oscillate $n_1 - 1$ and $n_2 - 1$ times respectively; the corresponding parameter values are $\nu_1 = (n_1 \pi)^2 P_1 - R_1$ and $\nu_1' = (n_2 \pi)^2 Q_1$. If we multiply (18) and (19) by $w(y)$ and $-v(x)$ respectively, add and set $u(x, y) = v(x)w(y)$ we obtain the differential equation

$$(P_1 u_x)_x - (Q_1 u_y)_y + [R_1 + (\nu_1 - \nu_1')]u = 0 \tag{20}$$

corresponding to (14) and for which explicit solutions $\sin n_1 \pi x \sin n_2 \pi y$ may be written down. The parameter value λ of (20) is then the difference $\nu_1 - \nu_1' = (n_1 \pi)^2 P_1 - R_1 - (n_2 \pi)^2 Q_1$ between the parameter values for the equations (18), (19), and is characterized by the number of oscillations $n_1 - 1$, $n_2 - 1$ of their solutions. Using the notation $\lambda_{n_1 n_2}$ they may be ordered as follows: $\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{31}, \lambda_{22}, \lambda_{13}, \cdots$. Corresponding to the first we have a solution $\sin \pi x \sin \pi y$ of (20) which does not oscillate, to the second $\sin 2\pi x \sin \pi y$ which oscillates once as a function of x, etc.

The solutions of the equation

$$k[P_2 u_{xx} - Q_2 u_{yy} + (R_2 + \lambda) u] = 0 \tag{21}$$

corresponding to (15) are again of the form $\sin n_1 \pi x \sin n_2 \pi y$ but now the characteristic number $\lambda_{n_1 n_2}$ is $(n_1 \pi)^2 P_2 - R_2 - (n_2 \pi)^2 Q_2$.

For any given n_1, n_2 and an arbitrarily chosen $\epsilon > 0$, we may choose $\overline{m}$, $\overline{\mu}$ such that the characteristic numbers of (16) and (17) differ from those of (18) and (19) by less than $\epsilon/2$ when $m \geqq \overline{m}$, $\mu \geqq \overline{\mu}$. Hence the characteristics $\lambda_{n_1 n_2}$ of (14) and (15) differ from those of (20) and (21) respectively by less than ϵ. There is then a parameter value of (A) which lies between

$$\pi^2 (n_1^2 P_1 - n_2^2 Q_1) - R_1 + \epsilon \quad \text{and} \quad \pi^2 (n_1^2 P_2 - n_2^2 Q_2) - R_2 - \epsilon.$$

The parameter value $\lambda_{n_1 n_2}$ of the general equation (A) shall be considered as that which is obtained by variation when the coefficients of (14) *or* (15) *are varied continuously in order to assume those of (A).*

The uniqueness of this determination of the λ's may be shown by the following argument. Let p, q, r be chosen as monotone functions of t in the interval $-1 \leqq t \leqq 1$ so that with increasing t the constant coefficients $P(t)$, $Q(t)$, $R(t)$ vary monotonely from $P_2 = P(1)$, $Q_2 = Q(1)$, $R_2 = R(1)$ to $P_1 = P(-1)$, $Q_1 = Q(-1)$, $R_1 = R(-1)$: the parameter $\lambda_{n_1 n_2}(t)$ will increase from the value $\lambda_{n_1 n_2}(t_2) = \pi^2 (n_1^2 P_2 - n_2^2 Q_2) - R_2$ to the value $\lambda_{n_1 n_2}(t_1) = \pi^2 (n_1^2 P_1 - n_2^2 Q_1) - R_1$. If the coefficients remained constants throughout, for each value of t the solution would be the product of a function of x alone oscillating $n_1 - 1$ times by a function of y alone oscillating $n_2 - 1$ times.

The value of the parameter will decrease if the coefficients vary monotonely in any fashion from kP_2, kQ_2, kR_2 to kP_1, kQ_1, kR_1 even if $p(t)$, $q(t)$, $r(t)$ do not remain constant throughout the region for intermediate values of the parameter t. If the values of the coefficients differ at most only infinitesimally from constants for any value of t, the value of the corresponding $\lambda_{n_1 n_2}$ will differ by at most an infinitesimal from the value $\lambda_{n_1 n_2}(t)$. By the introduction of another parameter s to take care of the variation of the coefficients from the original values $P(t)$, $Q(t)$, $R(t)$, we can have $\lambda_{n_1 n_2}(t, s)$

a decreasing function of t for all values of s and a continuous function of s for all values of t. But the equation (A) may be regarded as obtained in this manner by a monotone variation of the coefficients $p(x, y, t, s)$, $q(x, y, t, s)$, $r(x, y, t, s)$. Since each of the coefficients $p(1, s)$, $q(1, s)$, $r(1, s)$, $p(-1, s)$, $q(-1, s)$, $r(-1, s)$ is a constant for all values of s and since $\lambda_{n_1 n_2}(1, s)$, $\lambda_{n_1 n_2}(-1, s)$ are uniquely determined by n_1 and n_2 it follows that $\lambda_{n_1 n_2}(t, s)$ is completely determined in this manner.

THEOREM I. *For each value of m, μ there is a characteristic number $\lambda_{n_1 n_2}$ of the equation (A) determined by the oscillation numbers n_1, n_2 of the solution of* (14) *or* (15). *For fixed n_1, n_2 the totality of $\lambda_{n_1 n_2}$ lies within a finite interval, upper and lower bounds for which may be readily determined.*

Each set of $\lambda_{n_1 n_2}$'s will have at least one limit number, and we thus can pick out a denumerable set of limit values for the characteristic numbers. The bounds of each characteristic number so obtained may be readily calculated and definite statements made in regard to the limit points of the set for the elliptic and hyperbolic cases. *For the former ($p > 0$, $q < 0$), there is a limit point at positive infinity only; for the latter ($p > 0$, $q > 0$), there are limit points at both positive and negative infinity beside possibilities of more in the finite region.*

4. SOME AUXILIARY THEOREMS

For subsequent use we shall now derive some miscellaneous auxiliary theorems relative to maxima and minima of quadratic forms and integrals and to the limits of linear forms.

THEOREM II. *The value of the integral of $x(du/dx)^2\,dx$ along the straight line joining the points $P_1(0, M)$, $P_2(1, N)$ (N being less than M) is less than that along any other curve lying above the line and joining these points.*

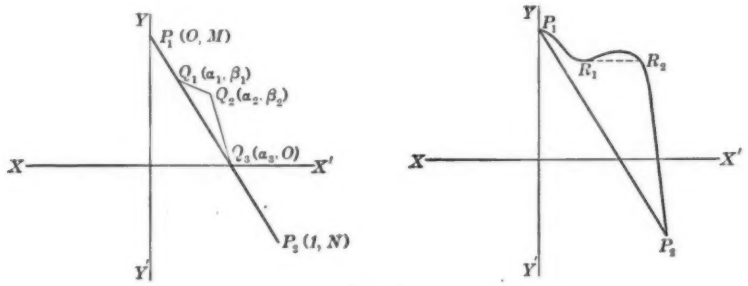

FIG. 1

To prove this we shall first show that the integral taken along two straight lines $Q_1 Q_2$, $Q_2 Q_3$ is greater than along $Q_1 Q_3$ (Fig. 1). Without loss of gen-

erality it may be assumed that the ordinate of Q_3 is 0. We wish to prove that

$$\int_{a_1}^{a_3} x\left(\frac{\alpha_3-\alpha_1}{\beta_1}\right)^2 dx < \int_{a_1}^{a_2} x\left(\frac{\alpha_2-\alpha_1}{\beta_2-\beta_1}\right)^2 dx + \int_{a_2}^{a_3} x\left(\frac{\alpha_3-\alpha_2}{\beta_2}\right)^2 dx$$

or

$$\frac{\alpha_3+\alpha_1}{\alpha_3-\alpha_1}\beta_1^2 < \frac{\alpha_2+\alpha_1}{\alpha_2-\alpha_1}(\beta_1-\beta_2)^2 + \frac{\alpha_2+\alpha_3}{\alpha_3-\alpha_2}\beta_2^2,$$

which may be written

$$\frac{\beta_1(\alpha_3-\alpha_2)}{\beta_2(\alpha_3-\alpha_1)}\alpha_1 + \frac{\beta_2(\alpha_3-\alpha_1)}{\beta_1(\alpha_3-\alpha_2)}\alpha_2 > \alpha_2+\alpha_1.$$

But if $\alpha_2 > \alpha_1 > 0$ and $0 < f < 1$ we know that $f\alpha_1 + \alpha_2/f > \alpha_2 + \alpha_1$. Hence our inequality is satisfied when $\beta_1/\beta_2 < (\alpha_3-\alpha_1)/(\alpha_3-\alpha_2)$, which certainly holds when Q_2 lies above the line $Q_1 Q_3$.

Since the integrand is never negative the integral along the dotted line $R_1 R_2$ whose slope is zero is less than that along the curved line $R_1 R_2$. In the discussion it is then necessary to use only monotone comparison curves. But such a function can be approximated as close as we wish by a monotone function composed of broken lines. By repeated use of the argument made above, our theorem is established.

Analogous to the division of the plane into rectangles as outlined in § 1, one can consider an interval 0, 1 in one dimension as being divided into m parts and at the points of division i/m $(i = 0, 1, \cdots, m)$ variables v_i as defined. Within the intervals the function v may be considered as made up of straight lines joining the points $(i/m, v_i)$. For convenience sake the intervals will be taken equal in length.

THEOREM III. *If within a subinterval σ/m, $(\sigma+\sigma')/m$ the oscillation of the v_i's is M, then $m\sum_{i=\sigma}^{\sigma+\sigma'}(v_i - v_{i-1})^2 > mM^2/\sigma'$.*

For, we have $m\sum_{i=\sigma}^{\sigma+\sigma'}(v_i - v_{i-1})^2 \geqq m/\sigma'\,[\sum_{i=\sigma}^{\sigma+\sigma'}|v_i - v_{i-1}|\,]^2$ and since $\sum_{i=\sigma}^{\sigma+\sigma'}|v_i - v_{i-1}|$ is at least equal to the oscillation, the theorem is established.

COROLLARY. *When $m\sum_i(v_i - v_{i-1})^2$ remains limited with increasing m, it is impossible that the v's have a limiting function $V(x)$ which is discontinuous.*

For, at a point of discontinuity of V let M be the oscillation. Given $\epsilon > 0$, it is possible to find an m such that in a subinterval of length $\sigma'/m < \epsilon$ the oscillation of the approximating function is greater than $M/2$. Then $m\sum_i(v_i - v_{i-1})^2 > M^2/4\epsilon$ which can be made great at pleasure. This being a contradiction, $V(x)$ must be continuous.

Returning now to the discussion of the two dimensional difference equations (A), we shall for the sake of convenience set $m = \mu$ and denote by A_m the maximum of the normalized solution for a given $\lambda_{n_1 n_2}$ and a given m.

Let A_m be taken at a point (σ, τ) $[u(\sigma, \tau) = A_m]$, and denote by $(\sigma + \sigma', \tau + \tau')$ the point furthest from (σ, τ) such that there is a chain of points connecting them for which $u_{ij} > A_m/2$.

THEOREM IV. *If A_m is the maximum* of those values u_{ij} for which*

$$\frac{1}{m^2}\sum_{ij} u_{ij}^2 = 1, \tag{22}$$

we have the formula

$$\frac{1}{m^2}\sum_{ij} m^2(u_{ij} - u_{i-1j})^2 + m^2(u_{ij} - u_{ij-1})^2 \geqq \frac{A_m^4\,\sigma'^2}{32^2\,m^2}. \tag{23}$$

In proving this it may without loss of generality be assumed† that $\sigma' \geqq \tau' \geqq 0$. From (22) it follows that the total number of points where $u_{ij} > A_m/4$ is less than $16m^2/A_m^2$. Let us select a rectangle with opposite vertices (σ, τ) and $(\sigma + \sigma', \tau + 32m^2/A_m^2\,\sigma')$ having double this number of points in it. On at least half the lines parallel to the y-axis in the rectangle, there is then a point where $u_{ij} \leqq A_m/4$ and on every such line a point where $u_{ij} \geqq A_m/2$. From Theorem III we have

$$\frac{1}{m^2}\sum_{ij} m^2(u_{ij} - u_{ij-1})^2 \geqq \frac{1}{2}\frac{\sigma'}{m}\sum_{j}\left(\frac{A_m}{4}\right)^2 \frac{A_m^2\,\sigma'}{32m} = \frac{A_m^4\,\sigma'^2}{32^2\,m^2},$$

from which (23) follows at once.

Were then

$$\lim_{m=\infty} \frac{A_m^2\,\sigma'}{m} = \infty,$$

the value of (23) would increase without limit as A_m increases. We can investigate the conditions under which

$$\lim_{m=\infty} \frac{A_m^2\,\sigma'}{m} \neq \infty.$$

Since in this case σ'/m is an infinitesimal of order at least as great as $1/A_m^2$, and since the same result would be obtained if we used fractional parts of A_m other than $A_m/2$, $A_m/4$, it follows that the point where u_{ij} increases indefinitely is isolated.

* A similar theorem can be obtained for the minimum by replacing u_{ij} by $-u_{ij}$.

† Were there not at least one such point on one of the four directions from (σ, τ) such that $u_{ij} > A_m/4$ we should have two particular terms, viz.,

$$\frac{1}{m^2}[m^2(u_{\sigma\tau} - u_{\sigma-1\tau})^2 + m^2(u_{\sigma\tau} - u_{\sigma\tau-1})^2]$$

of the sum (23) greater than $\frac{1}{2}A_m^2$ but since from (22) $A_m^2 < m^2$, it follows that this is greater than $A_m^4/2m^2$; this relation satisfies all the purposes of the theorem.

THEOREM V. *If at a point P a function u defined over a unit square R takes on a maximum M and if*

$$\int_R u^2\, dR = 1,$$

then

$$\int_R (u_x^2 + u_y^2)\, dR \geqq M^4 \sigma_M^2/32^2,$$

where σ_M is the radius of the greatest circle such that there is a point Q on the circumference which may be joined to P by a curve along which $u \geqq M/2$.

The proof of this theorem is analogous to that of Theorem IV.

Let us now deduce some properties of solutions of the elliptic partial differential equation which for the sake of simplicity may be taken in the form

$$u_{xx} + u_{yy} + (r + \lambda k)\, u = 0. \tag{24}$$

If this is multiplied by u and integrated over a field S bounded by a curve C there results the formula

$$\int_S (r + \lambda k)\, u^2\, dS = \int_S (u_x^2 + u_y^2)\, dS + \int_C u \frac{\partial u}{\partial n} ds,$$

where $\partial u/\partial n$ is the normal derivative directed inward; and hence

$$\max\,(r + \lambda k) \int_S u^2\, dS \geqq \int_S (u_x^2 + u_y^2)\, dS + \int_C u \frac{\partial u}{\partial n} ds. \tag{25}$$

Using the notation of Theorem V, let K denote a circular conical surface having for its axis the z-axis and cutting the plane $z = M/2$ in a circle of radius σ_M. The surface $u(x, y)$ lies outside the cone at the vertex and inside or on the surface at the section $z = M/2$. *Denoting by C_1 the projection on the xy-plane of the intersection of K and the surface $u(x, y)$ and by S that part of the region R in the xy-plane bounded by C_1, then*

$$\int_S (u_x^2 + u_y^2)\, dS > \frac{1}{4\sigma_M^2} \int_S u^2\, dS. \tag{26}$$

For, on passing to polar coördinates we have

$$\int_S (u_x^2 + u_y^2)\, dS = \int_S \left(u_\rho^2 + \frac{u_\theta^2}{\rho^2} \right) \rho d\rho d\theta \geqq \int_S u_\rho^2\, \rho d\rho d\theta. \tag{27}$$

From Theorem II we see that this last integral is greater than the value computed as if u were on the conical surface, in which case u_ρ would be $-M/2\sigma_M$. Hence from (27)

$$\int_S (u_x^2 + u_y^2)\, dS > \frac{M^2}{8\sigma_M^2} \int_0^{2\pi} \rho_{C_1}^2(\theta) d\theta, \tag{28}$$

where ρ_{C_1} is the value of ρ on the curve C_1. But since $u \leqq M$ it follows that

$$\int_S u^2\,dS \leqq M^2 \int_S dS = M^2 \int_S \rho d\rho d\theta = \frac{M^2}{2} \int_0^{2\pi} \rho_{C_1}^2(\theta)\,d\theta. \tag{29}$$

Formula (26) follows now from (28) and (29).

THEOREM VI. *The value of the integral*

$$\int_R (u_x^2 + u_y^2)\,dR$$

for a normalized solution $u(x, y)$ of the elliptic differential equation (24) *having a maximum M is greater than*

$$\frac{M^2}{64^2 \max (r + \lambda k)}.$$

For, on the curve C_1 the derivative $\partial u/\partial n$ is positive and hence the line integral in (25) is positive. Using formulas (25) and (26) we get

$$\begin{aligned}\int_S (u_x^2 + u_y^2)\,dS &< \max (r + \lambda k) \int_S u^2\,dS \\ &< 4\sigma_M^2 \max (r + \lambda k) \int_S (u_x^2 + u_y^2)\,dS,\end{aligned} \tag{30}$$

and hence

$$4\sigma_M^2 > \frac{1}{\max (r + \lambda k)}.$$

From this formula and Theorem V we have

$$\int_R (u_x^2 + u_y^2)\,dR > \frac{M^4 \sigma_M^2}{32^2} > \frac{M^4}{64^2 \max (r + \lambda k)}. \tag{31}$$

COROLLARY I. *The value of the integral $\int_R (u_x^2 + u_y^2)\,dR$ is greater than*

$$\frac{M^4}{64^2 \max (r + \lambda k) \int_R u^2\,dR}.$$

This may be seen by following through the steps of the reasoning when the hypothesis that the solutions are normalized is omitted.

COROLLARY II. *The maximum value of a normalized solution of the equation* (24) *is less than* $8\sqrt{\max (r + \lambda k)}$.

This follows from (31) and (25).

Let us next consider some properties of the hyperbolic equation and its approximating difference equations. For the sake of simplicity of notation

the equations will be taken in the canonical form

$$u_{xy} + (r + \lambda k)u = 0$$

and

$$m\mu(u_{ij} - u_{ij-1} - u_{i-1j} + u_{i-1j-1}) + (r_{ij} + \lambda k_{ij})u_{ij} = 0,$$

the solutions being supposed as before to vanish on the boundary of the region. The characteristics are now parallel to the axes and by a simple summation of the difference equations we get the formula

$$(32) \qquad [u(A) + u(C) - u(B) - u(D)] = \frac{1}{m\mu}\sum_{ij}(r_{ij} + \lambda k_{ij})u_{ij},$$

where A, B, C, D are the vertices of a rectangle with characteristics for sides, the summation extending over this rectangle.

THEOREM VII. *If for limited u_{ij} the single sum**

$$\frac{1}{m}\sum_{i} m^2(u_{ij} - u_{i-1j})^2$$

is unlimited for some section parallel to the x-axes it is unlimited for all sections and the double sum

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij} - u_{i-1j})^2$$

is also infinite.

For, if we square the special case of the formula (32)

$$m(u_{st} - u_{s-1t}) = m(u_{s\tau} - u_{s-1\tau}) - \frac{1}{\mu}\sum_{j=\tau}^{t}(r_{sj} + \lambda k_{sj})u_{sj},$$

where (s, t) is any lattice point and (s, τ) a point such that $t > \tau$, and then divide by m and sum over s, we obtain the relation

$$\sum_i m(u_{it} - u_{i-1t})^2 - \sum_i m(u_{i\tau} - u_{i-1\tau})^2 = \frac{1}{m}\sum_i f_i^2(r, k) + \sum_i 2f_i(r, k)(u_{i\tau} - u_{i-1\tau}),$$

where the totality of the functions

$$f_i(r, k) \equiv -\frac{1}{\mu}\sum_{j=\tau}^{t}(r_{sj} + \lambda k_{sj})u_{sj}$$

is limited, each being the average of a set of limited functions. The first sum on the right being the average of the f_i^2 is then limited. If the second sum is limited, the sums $\sum_i m(u_{it} - u_{i-1t})^2$ and $\sum_i m(u_{i\tau} - u_{i-1\tau})^2$ increase indefinitely together, their difference being finite. If the second sum is unlimited, its order of infinity is not more than half that of the sums on the

* Corresponding to the single integral $\int_0^1 u_x^2\,dx$.

left. For, we have

$$|\sum_i f_i(r, k)(u_{ir} - u_{i-1r})| < \max |f_i| \sum_i |u_{ir} - u_{i-1r}|,$$

$$\sum_i |u_{ir} - u_{i-1r}| \leqq \sqrt{m \sum_i (u_{ir} - u_{i-1r})^2}.$$

Hence the argument holds also for this case. And further if we sum over all values of t and multiply by $1/\mu$ we are again taking an average and the double sum

$$\frac{1}{m\mu} \sum_{ij} m^2 (u_{ij} - u_{i-1j})^2$$

approaches infinity.

Similar theorems may be readily established for the sums

$$\sum_j \mu (u_{ij} - u_{ij-1})^2 \qquad \text{and} \qquad \frac{1}{m\mu} \sum_{ij} \mu^2 (u_{ij} - u_{ij-1})^2.$$

In discussing the effect on the solutions of the equations (A) of increasing m, μ, let us consider the hyperbolic case (which presents the greater difficulty) and for simplicity of notation set $p_{ij} = q_{ij} = 1$ and $m = \mu$. Adding the equations for $u_{i+1j}, u_{i-1j}, u_{ij+1}, u_{ij-1}$ we have

$$(\alpha) \qquad \begin{aligned} &m^2 (u_{i+2j} + u_{i-2j} - u_{ij+2} - u_{ij-2}) + 4(r_{ij} + \lambda k_{ij}) u_{ij} \\ &= 4(r_{ij} + \lambda k_{ij}) u_{ij} - (r_{i+1j} + \lambda k_{i+1j}) u_{i+1j} - (r_{i-1j} + \lambda k_{i-1j}) u_{i-1j} \\ &\quad - (r_{ij+1} + \lambda k_{ij+1}) u_{ij+1} - (r_{ij-1} + \lambda k_{ij-1}) u_{ij-1}, \end{aligned}$$

while the set of equations for a value of m half the original is

$$(\beta) \qquad m^2 (u_{i+2j} + u_{i-2j} - u_{ij+2} - u_{ij-2}) + 4(r_{ij} + \lambda k_{ij}) u_{ij} = 0.$$

For each value of m the equations (A) will have a set of characteristic numbers λ. Let us choose a λ for some m and for larger values of m pick out (by the method given in § 3) λ's corresponding to this first one, the totality of such λ's being limited. On the assumption that r and k and their first difference quotients $m(r_{ij} - r_{i-1j})$, $m(r_{ij} - r_{ij-1})$, etc., are limited and that the values of u_{ij} and the first difference quotients $m(u_{ij} - u_{i-1j})$, $m(u_{ij} - u_{ij-1})$ are limited, there will then be an upper bound M for the absolute value of these functions and their difference quotients.

The left-hand members of equations (α), (β) are identical in form and we shall now show that *the right-hand side of* (α) *is in all cases an infinitesimal depending on* $1/m$. For, we note that the expression $r_{ij} u_{ij} - r_{i+1j} u_{i+1j}$ is in absolute value not greater than

$$|r_{i+1j} - r_{ij}| \, |u_{ij+1}| + |r_{ij}| \, |u_{ij+1} - u_{ij}| < \frac{2M^2}{m}.$$

We have also

$$|r_{ij}\,u_{ij} - r_{i-1j}\,u_{i-1j}| < \frac{2M^2}{m}, \qquad |r_{ij}\,u_{ij} - r_{ij+1}\,u_{ij+1}| < \frac{2M^2}{m},$$

$$|r_{ij}\,u_{ij} - r_{ij-1}\,u_{ij-1}| < \frac{2M^2}{m};$$

and similar relations for the four corresponding terms in k. Hence the right-hand side of (α) is less in absolute value than $8M^2/m\,[1 +$ upper bound of $|\lambda|\,]$ and is then an infinitesimal of order at least as high as $1/m$.

As is immediately evident, the same discussion would show that if m were multiplied by any power of 2 and the corresponding equations were added together in the proper manner, there would result a set of equations each differing from (β) only by terms on the right which taken together form an infinitesimal depending on $1/m$. Such a state of affairs would lead us to suspect that we can for the various values of m pick out corresponding solutions u_{ij} which are very close to one another and have one and only one limit function as m increases indefinitely.

5. The elliptic equations

In this section we shall consider the elliptic case of the difference equations (A) $(p > 0, q < 0)$. It will be found possible to trace for various values of m, μ the relations between the corresponding values of λ and u_{ij} for these equations. When m or μ is doubled, the number of equations and the number of characteristic values and solutions is approximately doubled. To be more exact, it is changed from $(m-1)(\mu-1)$ to $(2m-1)(\mu-1)$ or $(m-1)(2\mu-1)$. Some of the new characteristic numbers are greater and some are less than any of the original set, as one may see by studying the minimum problem. There are, however, $(m-1)(\mu-1)$ of the new set to each one of which there is related one of the old set in that the latter is a more or less rough approximation to it and the corresponding members of the set of u_{ij}'s are similarly related. If m and μ are further increased, one may trace the new λ's and u_{ij}'s which are related to those already selected. And these approach as limit the parameter values and solutions of the elliptic partial differential equation

$$(33) \qquad (pu_x)_x - (qu_y)_y + (r + \lambda k)u = 0, \qquad p > 0, \qquad q < 0.$$

The details of this discussion we now proceed to give.

We have already shown in § 3 that for fixed n_1, n_2 the set of parameter values is limited for all m, μ. From this it is possible to show that the various sets of u_{ij}'s corresponding to the $\lambda_{n_1 n_2}$'s are limited. With this end in view let us prove the following

THEOREM VIII. *If the maximum of the normalized solutions of (A) increases without limit with m, μ, the sum*

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2$$

increases without limit.

To prove this let us make use of the results of Theorem VII and its corollaries by passing a surface consisting of portions of planes through the points $(i/m, i/\mu, u_{ij})$. Let us agree to pass planes through the sets of three points corresponding to (i,j), $(i+1,j)$, $(i,j+1)$, and others through those corresponding to $(i+1,j)$, $(i,j+1)$, $(i+1,j+1)$ and compare the integrals

$$\int_R u^2\,dR,\ \int_R (u_x^2+u_y^2)\,dR$$

for the total surface found in this way with the sums

$$\frac{1}{m\mu}\sum_{ij} u_{ij}^2,\qquad \frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2$$

respectively.

As a first result let us show that the integral

$$\int_R (u_x^2+u_y^2)\,dR$$

is equivalent to

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2. \tag{34}$$

If we consider a triangle on the surface defined by the points (i,j), $(i-1,j)$, $(i,j-1)$ the slope in the x-direction is $m(u_{ij}-u_{i-1j})$ and in the y-direction is $\mu(u_{ij}-u_{ij-1})$. The integrand being a constant and the area being $1/2m\mu$, the integral has a value

$$\frac{1}{2m\mu}[\,m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2]. \tag{35}$$

For the triangle determined by (i,j), $(i-1,j)$, $(i-1,j+1)$, the integral has a value

$$\frac{1}{2m\mu}[\,m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{i-1j+1}-u_{i-1j})^2], \tag{36}$$

and for the triangle determined by (i,j), $(i,j-1)$, $(i+1,j-1)$, the integral has a value

$$\frac{1}{2m\mu}[\,m^2(u_{i+1j}-u_{ij})^2+\mu^2(u_{ij}-u_{ij-1})^2]. \tag{37}$$

By adding (35), the first part of (36) and the last part of (37) we get a term of (34) and similar considerations for the other triangles establish this first result.

As a second result let us show that the sum

$$\frac{1}{m\mu}\sum_{ij} u_{ij}^2$$

is greater than the integral

$$\frac{1}{9}\int_R u^2\,dR$$

where u is the surface consisting of planes. The integral of u^2 over one of the triangles is not greater than

$$\frac{1}{2m\mu}\max\,(u_{ij})^2$$

which is less than

$$\frac{1}{2m\mu}(\,|u_{ij}|+|u_{i+1j}|+|u_{ij+1}|\,)^2.$$

Summing over all the triangles we see that this is less than

$$\frac{9}{m\mu}\sum_{ij} u_{ij}^2.$$

From these two results and Theorem VI, Corollary I, we have the formula

$$\frac{1}{m\mu}\sum_{ij} m^2\,(u_{ij}-u_{i-1j})^2+\mu^2\,(u_{ij}-u_{ij-1})^2 > \frac{M^4}{192^2\,\max\,(r+\lambda k)\,\dfrac{1}{m\mu}\displaystyle\sum_{ij} u_{ij}^2},$$

where M is the maximum of u_{ij}. Theorem VIII is then established.

In formula (8) which when written in full is

$$\frac{1}{m\mu}\sum_{ij}[\,m^2\,p_{ij}\,(u_{ij}-u_{i-1j})^2-\mu^2\,q_{ij}\,(u_{ij}-u_{ij-1})^2] = \frac{1}{m\mu}\sum_{ij}(r_{ij}+\lambda k_{ij})\,u_{ij}^2, \tag{38}$$

let us consider normalized values of u_{ij}, $(\sum_{ij} k_{ij}\,u_{ij}^2 = m\mu)$, and note that since $\max\,|r_{ij}|/k_{ij}$ is finite and, since for any particular suite of $\lambda_{n_1n_2}$ on which attention is fixed the value of λ is limited, the right-hand member is limited for all m, μ. However, since p_{ij} and $-q_{ij}$ are greater than some positive constant, it follows from Theorem VIII that if $\max\,u_{ij}$ increases without limit the left-hand member of (38) increases without limit. This would establish a contradiction and hence the

THEOREM IX. *Those solutions of the elliptic case of equations (A) which correspond to a particular set of $\lambda_{n_1n_2}$ are limited for all m, μ.*

In proceeding to prove that the first difference quotient $m(u_{ij} - u_{i-1j})$ is limited, we shall confine* ourselves to the case where p and q are constants and denote them by π and ρ respectively. It will naturally be necessary to put some restrictions on the first difference quotients of the coefficients of the equations. *Let us assume that the first derivatives of r and k exist and are limited.* On differentiation of the equation (24) with regard to x one obtains

$$\pi u_{xxx} - \rho u_{xyy} + (ru + \lambda ku)_x = 0. \tag{39}$$

We note that $u = 0$ on the whole boundary of the square, $u_x = 0$ and $u_{xx} = 0$ on the sides $y = 0, y = 1$, and $u_y = 0$ and $u_{yy} = 0$ on the other sides. Hence for the special case $p = \pi, q = \rho$ it follows from (24) that $u_{xx} = 0$ and $u_{yy} = 0$ on the whole contour. If we multiply (39) by u_x and integrate over the square we note from the resulting relation

$$\int_R (\pi u_{xx}^2 - \rho u_{xy}^2)\, dR = \int_R [(r + \lambda k) u_x^2 + (r + \lambda k)_x u_x u]\, dR, \tag{40}$$

that if u_x is infinite the order of infinity of the left-hand integrand is higher by two than the right, which would lead us to suspect a contradiction.

With this process for differential equations as a guide let us set up by subtraction of the equations for i, j and $i - 1, j$ in (A) a new set of difference equations corresponding to (39). For the case $p(x, y) = \pi$, $q(x, y) = \rho$, after multiplication by m these become

$$\begin{aligned} & m^3 \pi [(u_{i+1j} - 2u_{ij} + u_{i-1j}) - (u_{ij} - 2u_{i-1j} + u_{i-2j})] \\ & - m\mu^2 \rho [(u_{ij+1} - 2u_{ij} + u_{ij-1}) - (u_{i-1j+1} - 2u_{i-1j} + u_{i-1j-1})] \\ & + m[(r_{ij} - r_{i-1j}) + \lambda(k_{ij} - k_{i-1j})] u_{i-1j} \\ & + m(r_{ij} + \lambda k_{ij})(u_{ij} - u_{i-1j}) = 0. \end{aligned} \tag{41}$$

On multiplication by $u_{ij} - u_{i-1j}$, summation over i, j, and division by μ we get

$$\begin{aligned} & \sum_{ij} \frac{m}{\mu} \pi [m(u_{i+1j} - u_{ij}) - m(u_{ij} - u_{i-1j})]^2 - \frac{\mu}{m} \rho [m(u_{ij+1} - u_{ij}) \\ & - m(u_{i-1j} - u_{i-1j-1})]^2 = \frac{m}{\mu} \sum_{ij} (r_{ij} + \lambda k_{ij})(u_{ij} - u_{i-1j})^2 \\ & + \frac{m}{\mu} \sum_{ij} [(r_{ij} - r_{i-1j}) + \lambda(k_{ij} - k_{i-1j})](u_{ij} - u_{i-1j}) u_{i-1j}. \end{aligned} \tag{42}$$

Having already shown that the definite quadratic form on the left of (38) is limited for all m, μ, it follows that

$$\frac{m}{\mu} \sum_{ij} (u_{ij} - u_{i-1j})^2$$

* This change is made here solely to avoid complicated notation.

is limited. From this we may deduce that since $r_{ij} + \lambda k_{ij}$ is limited, the first sum on the right of (42) is limited and further from the formula

$$m\mu \sum_{ij} (u_{ij} - u_{i-1j})^2 \equiv [\sum_{ij} |u_{ij} - u_{i-1j}|]^2,$$

or

$$\frac{m}{\mu} \sum_{ij} (u_{ij} - u_{i-1j})^2 \equiv \left[\frac{1}{\mu} \sum |u_{ij} - u_{i-1j}|\right]^2$$

that

$$\frac{1}{\mu} \sum_{ij} |u_{ij} - u_{i-1j}|$$

is limited. The second sum on the right of (42) being less in absolute value than

$$\max m|(r_{ij} - r_{i-1j}) + \lambda(k_{ij} - k_{i-1j})| \max |u_{i-1j}| \frac{1}{\mu} \sum_{ij} |u_{ij} - u_{i-1j}|,$$

must then remain limited for all m, μ.

The equation (39) may be written

$$(\pi u_x)_{xx} - (\rho u_x)_{yy} + (r + \lambda k) u_x = -(r_x + \lambda k_x) u$$

and regarded as a non-homogeneous equation of the second order in u_x. All that is essential in the reasoning of Theorems VI, VII, VIII may be applied to this equation and the corresponding difference equations when u and u_{ij} are limited. Regarding the difference quotient as unit instead of u_{ij} itself, it follows at once from a theorem analogous to VIII that the left-hand side of (42) increases without limit with $m(u_{ij} - u_{i-1j})$. This would lead to a contradiction. We have then

THEOREM X. *If the coefficients of the elliptic case of the difference equations (A) are values taken from functions which have continuous first derivatives, the first differences* $m(u_{ij} - u_{i-1j})$, $\mu(u_{ij} - u_{ij-1})$ *which correspond to a particular set of* $\lambda_{n_1 n_2}$ *are limited for all* m, μ.

By setting up difference equations by subtraction in (41) or (*A*) it may be proved in the same way that *the three second difference quotients corresponding to* u_{xx}, u_{xy}, u_{yy} *are limited.* To indicate briefly the method used in one of these cases let us select that corresponding to u_{xy} and confine ourselves to derivative notation, remembering that every step may be duplicated by difference equations. On differentiating (39) with regard to y we get

$$\pi u_{xyxx} - \rho u_{xyyy} + (ru + \lambda k u)_{xy} = 0 \tag{43}$$

and on multiplication by u_{xy} and integration, the formula*

$$\int_R [\pi u_{xyx}^2 - \rho u_{xyy}^2]\, dR = \int_R (r + \lambda k) u_{xy}^2\, dR + \int_R [(r + \lambda k)_x u_y + (r + \lambda k)_y u_x + (r + \lambda k)_{xy} u]\, u_{xy}\, dR. \tag{44}$$

* The vanishing of u_{xx} and u_{yy} on the contour entails the vanishing of u_{xxy} and u_{xyy} also.

Since the right-hand side of (40) is limited it follows that the left-hand side and hence that the integral of u_{xy}^2 is limited. The first integral on the right of (44) must then be limited and on the assumption that r_{xy}, k_{xy} are limited the second is *a fortiori* limited.* But were u_{xy} itself unlimited the integral on the left could be proved unlimited by an extension of Theorems VI, VII, VIII, where u_{xy} (and not u) is regarded as the unit. This would establish a contradiction.

And by an extension of the method, higher difference quotients can be shown limited if enough derivatives for r and k are assumed to exist.

From well-known theorems it follows that since u and its first difference quotients remain limited for all m, μ, *there is at least one limit function* $u(x, y)$, *which is approached uniformly by the solutions of the difference equations. On the assumption that the functions* p, q, r, k *have limited third derivatives, the function* u *also possesses continuous first and second derivatives.* On account of this continuity *the function* $u(x, y)$ *satisfying* (33) *for an everywhere dense set of points must throughout the region be a solution of this differential equation.*

From relation (8) it follows that each set of parameter values of the difference equations approaches the corresponding parameter value of the differential equation uniformly. The λ's are roots of an algebraic equation of order $(m-1)(\mu-1)$ and being continuous functions of its coefficients are continuous functions of the parameter t. Since the u_{ij}'s being roots of the linear algebraic equations (A) are continuous functions of the parameter t on which the coefficients p_{ij}, q_{ij}, r_{ij}, k_{ij}, and λ depend and, since they approach the function $u(x, y)$ uniformly, it follows that the limit function $u(x, y)$ is a continuous function of t. From this we can deduce that *there is only one limit for the set of* u_{ij}*'s.* For, in the case of equations (16) in one dimension with constant coefficients it may be assumed from the theory of ordinary differential equations that there is only one limit for each of the sets of v's and w's. For the case of the equation (14) in two dimensions with constant coefficients it follows from this immediately that the u_{ij}'s have one limit function only. In considering as in § 3 the coefficients of (A) as functions of t if there is a point t_0 for which there is a single solution $U_{n_1 n_2}(x, y)$ of (33) while for $t_0 + \epsilon$ there are two solutions $U_{n_1 n_2}(x, y)$, $\overline{U}_{n_1 n_2}(x, y)$ for every positive ϵ, we have as the limit of (9) the orthogonal relation

$$\int_R k U_{n_1 n_2}(t_0 + \epsilon, x, y)\, \overline{U}_{n_1 n_2}(t_0 + \epsilon, x, y)\, dR = 0,$$

which on passing to the limit $\epsilon = 0$ gives

* The function $r + \lambda k$ with its derivatives and the functions u, u_x, u_y being limited, the only inequality needed is $\int_R u_{xy}^2\, dR \geqq \left[\int_R |u_{xy}|\, dR\right]^2$.

$$\int_R kU^2_{n_1n_2}(t_0, x, y)\,dR = 0.$$

This is, however, impossible since the solutions are normalized and the result is established. We may now formulate

THEOREM XI. *There exists in the orthogonal elliptic case* ($q < 0$, $k > 0$) *a denumerably infinite set of parameter values* λ, *with limiting point at infinity only, for each of which there is a solution of* (33) *vanishing on the boundary.*

A similar fundamental theorem may be established for the polar case of (33), viz.; when $r(x, y) \leqq 0$ and the region R may be divided into a finite number of simple regions in the interior of some of which k is positive and in the interior of the remainder of which k is negative. The essential point of difference in the treatment of the polar case is in showing that the corresponding suite of $\lambda_{n_1n_2}$'s for the various values of m, μ is limited. This may be established by showing that: (*a*) the absolute value of all parameters $\lambda_{n_1n_2}$ of the equations (33) are increased by decreasing the size of the region and retaining the condition that u vanish on the boundary; (*b*) on taking for the smaller region a sub-rectangle in which $k > 0$, the value of each of the positive $\lambda_{n_1n_2}$'s is still further increased by replacing k by its minimum (> 0) in the sub-rectangle and r by its maximum. This gives an upper bound (which may be calculated explicitly) for each positive $\lambda_{n_1n_2}$; (*c*) on taking for the smaller region a sub-rectangle in which $k < 0$ the value of each of the negative $\lambda_{n_1n_2}$'s is still further decreased by replacing k by its maximum (< 0) in the sub-rectangle and r by its maximum. This gives a lower bound (which may be calculated explicitly) for each negative $\lambda_{n_1n_2}$.

6. THE HYPERBOLIC EQUATIONS

Preliminary to an investigation of some properties of the hyperbolic difference equations let us study some properties of the hyperbolic differential equation

$$(pu_x)_x - (qu_y)_y + (r + \lambda k)u = 0 \qquad (p > 0, q > 0). \tag{45}$$

While an analogous discussion can be given for the more general equation, we shall, for the sake of simplicity, confine ourselves to the case where p and q are constants. The characteristics will then be sets of straight lines making equal angles ($\gamma = \arctan \pm \sqrt{q/p}$) with the x-axis. It will be assumed that r and k are continuous with as many derivatives as desired and that the solutions of (45) are continuous and possess first and second derivatives.

THEOREM XII. *If the sides of a parallelogram ABCD are characteristics of the differential equation then*

$$2\sqrt{pq}\,[u(A) + u(C) - u(B) - u(D)] = \int_P (r + \lambda k)u\,dP, \tag{46}$$

the double integral being taken over the interior of the parallelogram. This formula holds identically for all such parallelograms and for all solutions of (45).

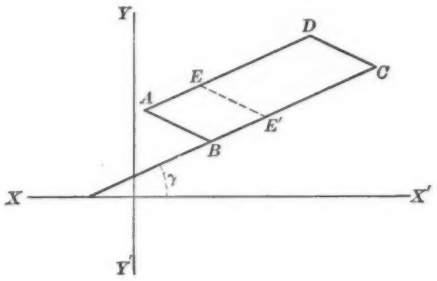

FIG. 2

For, on integrating the equation (45) over P we get

$$\int_\Gamma pu_x\,dy - qu_y\,dx = -\int_P (r+\lambda k)\,u dP,$$

where Γ is composed of the sides AB, BC, CD, DA. To calculate the line integral along AB, we resolve it into two parts, one along AB and the other normal to it. We have the formulas

$$dx = \sqrt{\frac{p}{p+q}}\,ds, \qquad dy = -\sqrt{\frac{q}{p+q}}\,ds,$$

$$u_x = -u_s\sqrt{\frac{p}{p+q}} - u_n\sqrt{\frac{q}{p+q}}, \qquad u_y = -u_s\sqrt{\frac{q}{p+q}} + u_n\sqrt{\frac{p}{p+q}};$$

and this part of the line integral becomes

$$\int_{AB} u_s\sqrt{pq}\,ds = \sqrt{pq}\,[u(B) - u(A)].$$

We have further

$$\int_{BC} pu_x\,dy - qu_y\,dx = \sqrt{pq}\,[u(B) - u(C)],$$

$$\int_{CD} pu_x\,dy - qu_y\,dx = \sqrt{pq}\,[u(D) - u(C)],$$

$$\int_{DA} pu_x\,dy - qu_y\,dx = \sqrt{pq}\,[u(D) - u(A)].$$

Formula (46) is then obtained by addition.

Let us make one or more of the following hypotheses.

HYPOTHESIS I. *The solutions of* (45) *are normalized; that is,* $\int_R ku^2\,dR = constant$, *where* R *is a unit square containing the parallelogram.*

If r and λ are limited the integral $\int_R |(r+\lambda k)u|\,dR$ will be limited. It may be assumed without loss of generality that this last integral is 1.

HYPOTHESIS II. *The integral* $\int_R (u_x^2+u_y^2)\,dR$ *is limited.*

Solutions which satisfy I cannot be infinite at a finite number of points in the region, as is readily seen from formula (46).

THEOREM XIII. *Solutions of the differential equation* (45) *which satisfy Hypotheses I and II are limited.*

For, let u take on a large value M at the point A. It follows from I that there will be points in the neighborhood of A where $u < M/2$. We may distinguish (Fig. 2) two cases: (*a*) $u(B) = u(D) = M/2$, while within the intervals AB and AD, $u > M/2$; (*b*) along at least one of the characteristics, $u > M/2$. In case (*a*) we have from (46) and Hypothesis I that $u(C) < 1$. Let η_1 and η_2 be lengths of AB, AD respectively, and $\eta_1 \leqq \eta_2$. For two corresponding points E and E' we have on applying formula (46) to the parallelogram $ABE'E$, the relation $u(E) - u(E') > M/2 - 1$. The double integral of $u_x^2 + u_y^2$ is independent of a rotation of axes and if the x-axis is taken in the direction AB we have from a theorem of integrals analogous to Theorem III the formula

$$\int_E^{E'} (u_x^2+u_y^2)\,dx \geqq \int_E^{E'} u_x^2\,dx \geqq \frac{\left(\frac{M}{2}-1\right)^2}{\eta_1}.$$

Hence

$$\int_P (u_x^2+u_y^2)\,dydx \geqq \frac{\left(\frac{M}{2}-1\right)^2}{\eta_1}\eta_2\cos 2\gamma \geqq \left(\frac{M}{2}-1\right)^2\cos 2\gamma,$$

and since this integral increases without limit with M, a contradiction establishes the theorem in this case.

Case (*b*) may be readily handled by the methods used in proving Theorem IV.

Instead of allowing u to increase indefinitely we can fix its maximum equal 1 and have a suite of functions u for which Hypotheses I and II become

HYPOTHESIS I′ $\qquad \lim \int_R u^2\,dx = 0,$

HYPOTHESIS II′ $\qquad \lim \int_R (u_x^2+u_y^2)\,dx = 0.$

From Theorem XII it follows at once that there can be no series of solutions of the differential equations having these properties.

Every solution of the differential equation (45) can be approximated as close as we wish in the parallelogram P by a solution of the difference equations; by defining a surface made up of portions of planes through the points u_{ij} one obtains from equation (A) the formula

$$(47) \quad 2\sqrt{pq}\,[u(A)+u(C)-u(B)-u(D)] = \frac{1}{m\mu}\sum_{ij}(r_{ij}+\lambda k_{ij})\,u_{ij}+\epsilon_{m\mu},$$

where the sum is taken over the lattice points within or on the parallelogram and $\epsilon_{m\mu}$ is an infinitesimal depending on m, μ.

One can, moreover, get a *precise* formula analogous to (46) for the difference equations when p, q are so chosen that the points of the lattice lie on the characteristics, that is, if $\sqrt{p/q}$ is a rational number. Let us derive such a formula for the case $p = q = 1$. The results in the more general case are similar. The difference equations for u_{ij} for the case $p = 1$, $q = 1$ are of the form

$$u_{i+1j} + u_{i-1j} - u_{ij+1} - u_{ij-1} = -1/m\mu\,(r_{ij} + \lambda k_{ij})\,u_{ij}.$$

Let us sum these over a rectangle whose vertices $A(\sigma_1, \tau_1)$, $B(\sigma_2, \tau_2)$, $C(\sigma_3, \tau_3)$, $D(\sigma_4, \tau_4)$ are at lattice points and whose sides make angles of $\pi/4$ with the axes. We note that for every point within the rectangle the term u_{ij} occurs four times in the sum: once each in a positive sense for those equations which enter for points next to the left and next to the right; once each in a negative sense for those next above and next below. This is also true for those points on the sides of the rectangles which are not vertices. For points lying in the next row below AB and to the left, with the exception of the end points $(\sigma_1 - 1, \tau_1)$, $(\sigma_2, \tau_2 - 1)$, the term occurs twice in the sum, once as positive and once as negative. A careful analysis of the remaining terms gives the formula

$$(48) \quad \begin{aligned}(u_{\sigma_1\tau_1} + u_{\sigma_1-1\tau_1}) + (u_{\sigma_2\tau_2} + u_{\sigma_2+1\tau_2}) - (u_{\sigma_2\tau_2} + u_{\sigma_2\tau_2-1}) \\ - (u_{\sigma_4\tau_4} + u_{\sigma_4\tau_4+1}) = 1/m\mu \sum_{ij}(r_{ij} + \lambda k_{ij})\,u_{ij}.\end{aligned}$$

Since under Hypothesis II the difference between the values of u at two neighboring points is less than a constant G, we have

$$(49) \quad \begin{aligned}2\,[u(A) + u(C) - u(B) - u(D)] = 1/m\mu \sum_{ij}(r_{ij} + \lambda k_{ij})\,u_{ij} + \eta_{m\mu}, \\ |\eta_{m\mu}| < 4G|.\end{aligned}$$

In general if $\sqrt{p/q}$ is rational we get a formula similar to (48).

THEOREM XIV. *When $\sqrt{p/q}$ is a rational number, while*

$$\frac{1}{m\mu}\sum_{ij} k_{ij}\,u_{ij}^2 = 1;$$

and

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2$$

is limited, it is impossible that max u_{ij} *increase indefinitely with* m, μ.

This is proved in a manner analogous to Theorem XII. While in the case where $\sqrt{p/q}$ is irrational precise formulas corresponding to (48) are not available, there seems to the author no reason to think that the theorem does not hold in that case. This theorem is based on formula (48) which depends in no way on boundary conditions but is identically true (when $\sqrt{p/q}$ is rational) for all solutions and all points of the lattice. Further, the solutions of the difference equations are continuous functions of the parameter t on which the coefficients p, q depend. From the principle of continuity we should expect a formula similar to (49) to obtain for all p and q.

As was seen in § 3, for each m, μ the limited function $\lambda_{n_1n_2}(t)$ is monotone in the interval $-1, 1$. Each function possesses a derivative and the points where $-d\lambda_{n_1n_2}/dt \geqq M > 0$ form an aggregate whose measure is less than c/M, c being a constant independent of M, m, μ. For the infinite set of functions $\lambda_{n_1n_2}$ corresponding to various values of m, μ there must exist at least one limit function and this function will be limited and monotone. It is well known that such a function possesses a limited derivative except possibly at a set of points of measure zero.

In formula (13)

$$\frac{d\lambda}{dt}=\frac{d\lambda}{dt}\frac{1}{m\mu}\sum_{ij}k_{ij}u_{ij}^2=-\frac{1}{m\mu}\sum_{ij}\frac{dr_{ij}}{dt}u_{ij}^2-\frac{\lambda}{m\mu}\sum_{ij}\frac{dk_{ij}}{dt}u_{ij}^2$$
$$+\frac{1}{m\mu}\sum_{ij}\left[m^2\frac{dp_{ij}}{dt}(u_{ij}-u_{i-1j})^2-\mu^2\frac{dq_{ij}}{dt}(u_{ij}-u_{ij-1})^2\right]$$

we may take (as in § 3) the limited derivatives $-dp/dt$, dq/dt, dr/dt, dk/dt positive and argue that the sum

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2$$

is limited for those values of t for which $d\lambda/dt$ is limited. It is then possible to enunciate the following

THEOREM XV. *For all values of* t *except possibly for a set of measure zero, the sum*

$$\frac{1}{m\mu}\sum_{ij} m^2(u_{ij}-u_{i-1j})^2+\mu^2(u_{ij}-u_{ij-1})^2$$

is limited.

If for a value of t for which $d\lambda/dt$ is finite a formula analogous to (48) is valid, we can deduce at once that u_{ij} remains limited for all m, μ and that

there is a limiting function $u(x, y)$ satisfying the differential equation. If, for example, for $t = t_0$ we have both $d\lambda/dt$ limited and $\sqrt{p/q}$ rational this would be the case.

From another point of view, any solution of the difference equations is an approximation to a solution of the differential equation. With increase of m, μ, a sequence of limited solutions of the difference equations which at a given lattice point are made to take on a value 1 will have at least one limiting function which is a solution of the differential equation at that point. But, as was seen above (cf. Hypotheses I′, II′), for any sequence of solutions of the differential equation it is impossible that

$$\lim \int_R u^2 \, dx = 0, \qquad \lim \int_R (u_x^2 + u_y^2) \, dx = 0.$$

But this seems to imply that for limited $d\lambda/dt$, normalized solutions of the difference equations like normalized solutions of (45) remain limited.

It may be proved by a method similar to that used in Theorem VII that when the double sum

$$\frac{1}{m\mu} \sum_{ij} m^2 \frac{dp_{ij}}{dt} (u_{ij} - u_{i-1j})^2 - \mu^2 \frac{dq_{ij}}{dt} (u_{ij} - u_{ij-1})^2$$

remains limited as m, μ increase, the single sums $\sum_i m(u_{ij} - u_{i-1j})^2$ for all j and $\sum_j \mu(u_{ij} - u_{ij-1})^2$ for all i remain limited. It follows from the corollary to Theorem III that the limiting function $u(x, y)$ of the sets of solutions of the difference equations (A) is a continuous function for all lines parallel to the axes which belong to the lattice for any m, μ and hence is continuous throughout the square.

If we make the hypothesis (which appears a most reasonable one) that a formula analogous to (48) can be obtained for any set of hyperbolic difference equations (A), there would be for all values of t (except perhaps for a set of measure zero) a bounded limiting continuous function $u(x, y)$ which would be a solution of (45). If there were a value of t for which such a function u did not exist, there would be coefficients as near to the original as desired such that for these the differential equation would have a solution.

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NOTE SUPPLEMENTARY TO THE PAPER "ON CERTAIN PAIRS OF TRANSCENDENTAL FUNCTIONS WHOSE ROOTS SEPARATE EACH OTHER"

BY

MAXIME BÔCHER

In the paper with the above title, which appeared in these Transactions, vol. 2 (1901), p. 428, I obtained a number of general theorems concerning the zeros of functions of the form $\phi_2 y' - \phi_1 y$, where y is a solution of a homogeneous linear differential equation of the second order. There is a further general theorem of the same character which I should have included in that paper if I had discovered it at the time, and which I now give as VII′ below, it being a counterpart to VII in the earlier paper. I also give three special applications of this general result, numbered VIII′, VIII″, VIII‴, since they are counterparts of VIII. A part of VIII′ was obtained by Fite in the Annals of Mathematics for June, 1917, (see the closing lines of his article) by another method; and it was this special result of Fite's that suggested to me the much more general Theorem VII′.

What follows should be regarded as an addendum to the former paper, to be inserted on page 434 at the end of § 2. All references are to that paper, to which the reader should turn for an explanation of the notation and a statement of the restrictions placed on the functions.

VII′. *If none of the six functions*

$$\phi_1\psi_2 - \phi_2\psi_1,\ \psi_1\chi_2 - \psi_2\chi_1,\ \chi_1\phi_2 - \chi_2\phi_1,\ \{\phi_1, \phi_2\},\ \{\psi_1, \psi_2\},\ \{\chi_1, \chi_2\}$$

vanishes in (I), *and if the last of them has the opposite sign from the two preceding ones, then no one of the functions* Φ, Ψ, $\mathbf{X}$ *can vanish more than once in* (I)*: and no two of them can vanish except perhaps* Φ *and* Ψ. *If these last two functions do both vanish in* (I), *the root of* Φ *will be greater or less than the root of* Ψ *according as the product of the six functions written above is negative or positive.*

The first part of this theorem is an immediate consequence of IV, p. 431. Let us then assume that Φ vanishes at x_1, Ψ at x_2. These two points cannot coincide, since then $\phi_1\psi_2 - \phi_2\psi_1$ would vanish at this point.

Suppose, first, that $x_1 < x_2$. Then

$$\operatorname{sgn} \Phi(x_2) = \operatorname{sgn} \Phi'(x_1),$$

and, consequently, by (3), p. 430,

$$\operatorname{sgn} \Phi(x_2) = -\operatorname{sgn} (\phi_1 \psi_2 - \phi_2 \psi_1) \operatorname{sgn} \{\phi_1, \phi_2\} \operatorname{sgn} \Psi(x_1).$$

On the other hand, if we substitute in the identity (6), p. 432, first $x = x_1$, then $x = x_2$, we find

$$\operatorname{sgn} (\phi_1 \psi_2 - \phi_2 \psi_1) \operatorname{sgn} \mathrm{X}(x_1) = -\operatorname{sgn} (\chi_1 \phi_2 - \chi_2 \phi_1) \operatorname{sgn} \Psi(x_1),$$

$$\operatorname{sgn} (\phi_1 \psi_2 - \phi_2 \psi_1) \operatorname{sgn} \mathrm{X}(x_2) = -\operatorname{sgn} (\psi_1 \chi_2 - \psi_2 \chi_1) \operatorname{sgn} \Phi(x_2).$$

Multiplying together the last three equations, and remembering that X does not vanish, so that

$$\operatorname{sgn} \mathrm{X}(x_1) = \operatorname{sgn} \mathrm{X}(x_2) = \pm 1,$$

we find

$$\operatorname{sgn} \Phi(x_2) = -\operatorname{sgn} [(\phi_1 \psi_2 - \phi_2 \psi_1)(\chi_1 \phi_2 - \chi_2 \phi_1)(\psi_1 \chi_2 - \psi_2 \chi_1)] \operatorname{sgn} \{\phi_1, \phi_2\} \operatorname{sgn} \Phi(x_2).$$

If here we cancel the factor $\operatorname{sgn} \Phi(x_2) = \pm 1$ from both sides, and replace -1 by its value $\operatorname{sgn} \{\psi_1, \psi_2\} \operatorname{sgn} \{\chi_1, \chi_2\}$, we infer from this equation that the product of the six functions of VII′ is positive.

In a precisely similar manner, we show that if $x_1 > x_2$, the product of these six factors is negative, and this completes the proof of VII′.

The theorem just proved admits of extensions in two directions. A brief mention of these will be sufficient.

First, the assumption that $\{\phi_1, \phi_2\}$, $\{\psi_1, \psi_2\}$, $\{\chi_1, \chi_2\}$ do not vanish may be replaced (at least if we assume that $\phi_1, \phi_2, \psi_1, \psi_2, \chi_1, \chi_2$ have *continuous* first derivatives) by the less restrictive conditions that these three functions do not change sign and that none of them vanishes at all points of a connected subinterval of (I). This extension is effected as in similar cases on p. 436.

Secondly, we may suppose that Φ, instead of being of the form $\phi_2 y' - \phi_1 y$, is of the form $\phi_1 y - \phi_2 z$, where (y, z) is a solution of a pair of homogeneous linear differential equations of the first order. Making corresponding changes in the definitions of Ψ and X, we obtain in this way a counterpart to XVI in these Transactions, vol. 3 (1902), p. 207, the proof being closely parallel to that we have here given.

We make a first application of VII′ by letting

$$\Phi = y, \qquad \Psi = y', \qquad \mathrm{X} = py' + qy = -y'',$$

so that

$$\phi_1\psi_2 - \phi_2\psi_1 = -1, \qquad \psi_1\chi_2 - \psi_2\chi_1 = q, \qquad \chi_1\phi_2 - \chi_2\phi_1 = p,$$

$$\{\phi_1, \phi_2\} = 1, \qquad \{\psi_1, \psi_2\} = q, \qquad \{\chi_1, \chi_2\} = q^2\left[\frac{d}{dx}\left(\frac{p}{q}\right) + 1\right].$$

Theorem VII′ now becomes*

VIII′. *If*

$$q > 0, \qquad \frac{d}{dx}\left(\frac{p}{q}\right) < -1, \qquad p \text{ nowhere zero},$$

then no one of the functions y, y', y'' can vanish more than once; and no two of them can vanish except y and y'. If these two functions do both vanish, the root of y is greater or less than the root of y' according as p is negative or positive.

As a second application of VII′, we let

$$X = y, \qquad \Psi = y', \qquad \Phi = -y''$$

and thus find

VIII″. *If*

$$q < 0, \qquad \frac{d}{dx}\left(\frac{p}{q}\right) < -1, \qquad p \text{ nowhere zero},$$

then no one of the functions y, y', y'' can vanish more than once; and no two of them can vanish except y' and y''. If these two functions do both vanish, the root of y' is greater or less than the root of y'' according as p is negative or positive.

Finally, letting

$$\Phi = y, \qquad X = y', \qquad \Psi = -y'',$$

we obtain

VIII‴. *If*

$$q < 0, \qquad \frac{d}{dx}\left(\frac{p}{q}\right) > -1, \qquad p \text{ nowhere zero},$$

then no one of the functions y, y', y'' can vanish more than once; and no two of them can vanish except y and y''. If these two functions do both vanish, the root of y is greater or less than the root of y'' according as p is negative or positive.

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July, 1917

* The latter part of this theorem is the result of Fite above referred to, except that Fite requires merely

$$\frac{d}{dx}\left(\frac{p}{q}\right) \leqq -1.$$

This generalization, however, is not a substantial one, since the equality sign may be removed by an arbitrarily small variation of p, and such a variation, if the root of y' is held fixed, produces only a slight change in the root of y.

A THEOREM FOR SPACE ANALOGOUS TO CESÀRO'S THEOREM FOR PLANE ISOGONAL SYSTEMS*

BY

PERCEY F. SMITH

Investigations† in the geometry of ordinary differential equations of the first order have yielded many interesting results, especially in the properties of the osculating circles of the integral curves of such equations. In particular, Scheffers (l. c.) has introduced two simple lineal element transformations, which have been well described by Kasner‡ as "turns" and "slides," respectively. A *turn* converts a lineal element into one having the same point and a direction making a given angle with the original direction. Under a *slide* an element is displaced a given distance along the line of the element. Applied to the integral curves of

$$F(x, y, p) = 0, \tag{1}$$

a turn of each lineal element satisfying the equation leads to a second equation whose integral curves form a system of isogonal trajectories of the integral curves of (1). A slide leads to equitangential trajectories. Cesàro§ has shown that the osculating circles of the curves of an isogonal system at any given point will pass through a second common point. Scheffers (l. c.) has demonstrated a companion theorem for an equitangential system of curves, namely, the osculating circles of the curves of such a system which touch a given line will also touch a second line.

In approaching questions of this nature for space, it is clear that an understanding must be reached as to the direction the investigation should take.‖

* Presented to the Society, December 2, 1916.

† The following papers may be referred to in this connection: Scheffers, *Isogonalkurven, Äquitangentialkurven und complexe Zahlen*, Mathematische Annalen, vol. 60 (1905), p. 491; Gundelfinger, *On the geometry of line elements in the plane with reference to osculating circles*, American Journal of Mathematics, vol. 33 (1910), p. 153.

‡ Kasner, *The group of turns and slides*, etc., American Journal of Mathematics, vol. 33 (1910), p. 193.

§ *Geometria intrinseca*, 1896.

‖ Kasner, for example, has proposed and solved a problem for space, different from that discussed here, in his paper *Equitangential congruences of curves in space*, Rendiconti del Circolo Matematico di Palermo, vol. 35 (1913), p. 283.

To consider partial differential equations of the first order and their integral surfaces is a first thought, but to consider osculating spheres of such surfaces as analogous to the osculating circles of plane curves is wide of the mark, since at a general point on a surface there is no osculating sphere. Furthermore, a turn, or a slide, of a lineal element is definite in so far as the point, or the line, of the element is unique. It is natural to define a *turn of a surface element as a rotation of the element through a given angle about any axis through the point of the element and lying on its plane.* Likewise by a *slide* we may understand *a translation of the element in its plane a given distance in any direction.* But in the first instance, the axis of the turn, and in the second, the direction of the slide, are not definite. A study of the geometry of partial differential equations of the first order, to which this paper is devoted, clears up the vagueness referred to above, and leads to theorems which appear to be as fundamental in connection with such differential equations as those cited for ordinary differential equations.

1. Preliminary formulas

Consider a surface element (x, y, z, p, q), and a consecutive element $(x + dx, y + dy, z + dz, p + dp, q + dq)$ *united* with the first, that is, such that

$$dz = p\,dx + q\,dy, \tag{1}$$

the ratios of the infinitesimals* satisfying

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{p\alpha + q\beta} = \frac{dp}{\delta} = \frac{dq}{\epsilon}. \tag{2}$$

The pair of united elements will establish ∞^1 curvature elements (x, y, z, p, q, r, s, t), where the last three coördinates satisfy

$$\delta = r\alpha + s\beta, \qquad \epsilon = s\alpha + t\beta. \tag{3}$$

For any surface to which the pair of united elements belong, the curvature of normal sections at (x, y, z) containing the direction $\delta x : \delta y : \delta z$ is

$$\frac{1}{R} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{r\,\delta x^2 + 2s\,\delta x\,\delta y + t\,\delta y^2}{\delta s^2}. \tag{4}$$

Eliminating r and s by using (3), this becomes

$$\frac{1}{R} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{\beta\delta\,\delta x^2 + \alpha\epsilon\,\delta y^2 - s(\beta\,\delta x - \alpha\,\delta y)^2}{\alpha\beta\,\delta s^2}. \tag{5}$$

* Cf. a paper by the author *On osculating element-bands associated with loci of surface elements*, these Transactions, vol. 11 (1910), p. 302.

The pair of surface elements under discussion present two directions (usually distinct) which possess obvious geometrical interest and simplicity. These directions establish a line-pair lying in the plane E of the first element (x, y, z, p, q) and passing through its point $P(x, y, z)$. The first direction is that of the line joining the points of the united elements. Here

$$\delta x : \delta y : \delta z :: \alpha : \beta : p\alpha + q\beta . \tag{6}$$

The second direction is that of the line of intersection of the planes of the elements. For this line

$$\delta x : \delta y : \delta z = \epsilon : -\delta : p\epsilon - q\delta . \tag{7}$$

For the first direction, equation (5) becomes

$$\frac{1}{R_1} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{\alpha\delta + \beta\epsilon}{\alpha^2 + \beta^2 + (p\alpha + q\beta)^2}. \tag{8}$$

The disappearance of s indicates that the normal curvature of every surface containing the pair of elements is the same in the direction in which these elements are united. The normal sections of these surfaces in this direction have therefore a common osculating circle. Hence we may associate with a pair of united surface elements a *first osculating circle* whose radius is given by (8) and which has the property described in the preceding sentences, namely, *it is the osculating circle of the normal section of any surface containing the element-pair taken in the direction in which the elements are united.*

Reserving for a later section (§ 3) a similar discussion for the direction (7), pass now to the application of the preceding exposition to the partial differential equation (assumed not linear)

$$F(x, y, z, p, q) = 0, \tag{9}$$

in which, as usual, $p = \partial z/\partial x$, $q = \partial z/\partial y$. Let S be an integral surface of this differential equation, $P(x, y, z)$ a point on S, and C the characteristic on S through P. Infinitely many integral surfaces of (9) contain the characteristic C and all touch S along C. On all of these surfaces the direction conjugate to that of the characteristic at P is common.* That is, in each surface element satisfying (9) there are two significant directions, namely, (1) *the direction of the characteristic determined by this element,* and (2), *the common conjugate direction* above described. If we consider a pair of united elements satisfying (9) whose points are on the characteristic determined by the first element, these two directions are obviously precisely those dis-

* Carathéodory, *Zur geometrischen Deutung der Charakteristiken einer partiellen Differentialgleichung erster Ordnung,* Mathematische Annalen, vol. 95 (1904), p. 377. An excellent exposition of the geometry involved here is given by Goursat in his *Leçons sur l'intégration des équations aux derivées partielles du premier ordre* (Paris, 1891), pp. 181–88.

cussed above (equations (6) and (7)). For this pair of elements, the infinitesimals dx, dy, etc., satisfy the well-known differential equations of the characteristics (or characteristic bands)

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-F_x - pF_z} = \frac{dq}{-F_y - qF_z}, \tag{10}$$

where subscripts indicate partial differentiation with respect to the variables expressed.

To obtain a *first osculating circle* for any surface element satisfying (9), we may use formula (8) in which α, β, etc., of equations (2) are to be replaced by the corresponding denominators in (10). The property possessed by this circle is clear. *The normal section of any integral surface containing this surface element taken in the direction of the characteristic will be osculated by this circle.* In the following discussion the phrase "first osculating circle for the surface element" will be employed to designate this circle.

To derive a convenient expression for the radius of the first osculating circle, it is desirable to introduce "homogeneous element coördinates" by setting

$$p = -\frac{p_1}{p_3}, \qquad q = -\frac{p_2}{p_3}. \tag{11}$$

The given differential equation (9) now becomes

$$F(x, y, z, p, q) = G(x, y, z, p_1, p_2, p_3) = 0, \tag{12}$$

where G is homogeneous and of degree zero in p_1, p_2, p_3. Hence, if we set $G_i = \partial G/\partial p_i$ $(i = 1, 2, 3)$,

$$p_1 G_1 + p_2 G_2 + p_3 G_3 = 0. \tag{13}$$

Furthermore, $F_p = -p_3 G_1$, $F_q = -p_3 G_2$, $pF_p + qF_q = -p_3 G_3$. Then we easily obtain from (8), (2), and (10) the symmetrical formula* for the *curvature of the first osculating circle*

$$\frac{1}{R_1} = \frac{G_1 G_x + G_2 G_y + G_3 G_z}{\omega(G_1^2 + G_2^2 + G_3^2)}, \tag{14}$$

where $\omega^2 = p_1^2 + p_2^2 + p_3^2$.

This convenient formula being secured we proceed in the following section

* It is not without interest to derive an expression for the curvature of the osculating circle at (x, y) of the integral curve of the ordinary differential equation $F(x, y, p) = 0$ determined by a lineal element (x, y, p) satisfying this equation. In fact, if we place $p = -p_1/p_2$, and proceed from the usual expression for curvature, it is easy to show that the resulting formula is precisely (14) written for two dimensions instead of three. In other words, *the first osculating circle here introduced is a literal generalization to three dimensions of the osculating circle of the plane.*

to apply to each of the surface elements of equation (12) the transformation designated a *turn*.

In the analytical developments which follow it will be necessary to refer to a result which may be assumed from differential geometry.

If $(\alpha_1, \alpha_2, \alpha_3)$, (l_1, l_2, l_3), and $(\lambda_1, \lambda_2, \lambda_3)$ are the direction cosines of three lines mutually at right angles, and if such directions are chosen on these lines that

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ l_1 & l_2 & l_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} = +1 \tag{14'}$$

(rather than the alternative value -1), then, indicating by primes differentiation with respect to a common parameter, relations will hold of which the following is the type:

$$\alpha_i' = l_i \begin{vmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} - \lambda_i \begin{vmatrix} l_1 & l_2 & l_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{vmatrix}. \tag{15}$$

2. Theorems for an isogonal system

Turn now to the differential equation

$$G(x, y, z, p_1, p_2, p_3) = 0, \tag{16}$$

and consider one of its surface elements (x, y, z, p_1, p_2, p_3). We give to the normal (λ) of this element a definite direction by choosing its direction cosines $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 = p_1/\omega, \quad \lambda_2 = p_2/\omega, \quad \lambda_3 = p_3/\omega, \quad \omega = +\sqrt{p_1^2 + p_2^2 + p_3^2}. \tag{17}$$

To the tangent (l) of the characteristic through (x, y, z) in this element we assign as positive direction that whose direction cosines are (see (10))

$$l_1 = G_1/G, \quad l_2 = G_2/G, \quad l_3 = G_3/G, \quad G = +\sqrt{G_1^2 + G_2^2 + G_3^2}. \tag{18}$$

For $F_p : F_q : pF_p + qF_q = G_1 : G_2 : G_3$.

We next establish an *axis* (α) through (x, y, z) in the plane of the element at right angles to (l) with direction cosines $(\alpha_1, \alpha_2, \alpha_3)$ and such that (14') holds. A rotation of the element about the axis (α) is now entirely definite, and we may set down the

Definition. *A turn of a surface element satisfying a partial differential equation of the first order is a rotation through a given angle about the axis* (α).

The equations for a turn T_θ through an angle θ are obviously,

$$\lambda_i' = \lambda_i \cos\theta + l_i \sin\theta \qquad (i = 1, 2, 3), \tag{19}$$

in which the direction cosines of the normal to the element in its new position

are $(\lambda_1', \lambda_2', \lambda_3')$. The turned elements will also satisfy a partial differential equation of the first order. To obtain this equation we replace λ_i and l_i in (19) by their values as defined in (17) and (18), and from the three equations thus obtained and (16) eliminate p_1, p_2, and p_3. Then placing

$$\lambda_i' = p_i'/\omega', \qquad \omega' = +\sqrt{p_1'^2 + p_2'^2 + p_3'^2} \qquad (i = 1, 2, 3), \tag{20}$$

the result of the elimination may be indicated by

$$G'(x, y, z, p_1', p_2', p_3') = 0, \tag{21}$$

in which the parameter θ is implicitly involved. Thus equation (21) represents a system of differential equations, which will be called an *isogonal system*. Justification for this terminology will appear presently.

The capital fact in connection with the turn T_θ given by (19) is this. *For a fixed point (x, y, z) the turns form a group.* In other words, *the direction of the characteristic in a turned element is perpendicular to the axis (α).* To establish the truth of this statement the following considerations may suffice. The planes of the surface elements of the differential equation (16) with a common point P envelop a cone,*—the *elementary cone* of Monge. The elements of this cone are the tangents of the characteristics through P. A turn converts the tangent planes of this cone into those of the elementary cone of the transformed equation. *The elements of the original cone also "turn" into those of the transformed cone.* This simple geometric fact is not common knowledge but may readily be established.

An important consequence of the preceding result is the following. If (l_1', l_2', l_3') are the direction cosines of the characteristic of $G' = 0$ in the turned element $(x, y, z, p_1', p_2', p_3')$, then

$$l_i' = -\lambda_i \sin\theta + l_i \cos\theta \qquad (i = 1, 2, 3), \tag{22}$$

since this characteristic is at right angles to the axis. Here $l_i' = G_i'/G'$, and $G'^2 = \sum_{i=1}^{i=3} G_i'^2$.

It will now be possible to prove for the isogonal system (21) a theorem analogous to that of Cesàro for plane isogonal systems. The preceding presentation should have made clear the following situation.

(I) Turning an element of the original partial differential equation $G = 0$ gives rise in the isogonal system $G' = 0$ to a pencil of elements whose axis is the axis of the turn.

(II) The characteristics in this pencil of elements are at right angles to the axis.

(III) The first osculating circles for the elements of this pencil are coplanar,

* For simplicity of statement, the equation is assumed not linear. The conclusion, however, is true also for this case.

their common plane being at right angles to the axis. Moreover, these circles have by definition one common point,—the common point of the pencil of elements.

To show that these osculating circles have a *second* common point is the next step. For this purpose, apply formula (14) to the curvature of the first osculating circle in the isogonal system (21). This leads to

$$\frac{1}{R_1'} = \frac{G_1' G_x' + G_2' G_y' + G_3' G_z'}{\omega' (G_1'^2 + G_2'^2 + G_3'^2)}. \tag{23}$$

Making the convenient and permissible assumption,

$$\omega = \omega', \tag{24}$$

the transformation T_θ (19) becomes

$$p_i' = p_i \cos\theta + l_i \omega \sin\theta. \tag{25}$$

The partial derivatives of $G = 0$ and the transformed equation $G' = 0$ are related as follows. First,

$$G_x = G_x' + \sin\theta\omega \sum_{i=1}^{3} G_i' \frac{\partial l_i}{\partial x}. \tag{26}$$

Now, by (22),

$$\frac{G_i'}{G'} = -\frac{p_i}{\omega} \sin\theta + l_i \cos\theta. \tag{27}$$

In consequence, the sum in (26) is

$$G' \left(-\frac{1}{\omega} \sin\theta \sum_i p_i \frac{\partial l_i}{\partial x} + \cos\theta \sum_i l_i \frac{\partial l_i}{\partial x} \right). \tag{28}$$

From $\sum p_i l_i = 0$, and $\sum l_i^2 = 1$, this expression vanishes. Hence

$$G_x = G_x', \qquad G_y = G_y', \qquad G_z = G_z'. \tag{29}$$

Again,

$$G_1 = G_1' \cos\theta + \sin\theta \sum_{i=1}^{3} G_i' \frac{\partial}{\partial p_1} (\omega l_i),$$

$$= G_1' \cos\theta + \frac{p_1}{\omega} \sin\theta \sum G_i' l_i + \omega \sin\theta \sum G_i' \frac{\partial l_i}{\partial p_1},$$

$$= G_1' \cos\theta + \frac{G' p_1}{\omega} \sin\theta \cos\theta + G' l_1 \sin^2\theta,$$

the last term arising from the fact that, since $\sum p_i l_i = 0$, then

$$\sum p_i \frac{\partial l_i}{\partial p_1} = -l_1.$$

Hence, referring to (25), the final result is

$$G_1 = G_1' \cos\theta + G' \frac{p_1'}{\omega} \sin\theta, \tag{29'}$$

with similar expressions for G_2 and G_3.

From these equations we find at once

$$\sum_i G_i^2 = \sum_i G_i'^2. \tag{30}$$

Thus the relations established in (27), (29), and (30), make it possible to write formula (23) in the form

$$\frac{1}{R_1'} = \frac{(G_1 G_x)}{\omega G^2} \cos\theta - \frac{(p_1 G_x)}{\omega^2 G} \sin\theta, \tag{31}$$

where

$$(G_1 G_x) = G_1 G_x + G_2 G_y + G_3 G_z, \qquad (p_1 G_x) = p_1 G_x + p_2 G_y + p_3 G_z.$$

From (31) follows the fact that the centers of the first osculating circles lie on a line. To see this it suffices to take as axes of rectangular coördinates the normals to the elements for $\theta = 0$, and $\theta = 90°$, and to remember that $R_1' \cos\theta$, $R_1' \sin\theta$ are the rectangular coördinates of the center of the osculating circle with respect to these axes. Hence is established

THEOREM 1. *When a surface element of a partial differential equation of the first order is turned about an axis in its plane drawn at right angles to the characteristic, the first osculating circles for the transformed elements form a coaxal system.*

The system of transformed equations

$$G'(x, y, z, p_1', p_2', p_3') = 0 \tag{21}$$

may properly be called as *isogonal system* from the following considerations. Assume S to be an integral surface of the original equation $G = 0$ (i. e., $\theta = 0$). Let us direct our attention upon an orthogonal trajectory D of the characteristics upon the surface S. Under T_θ each surface element of S along the curve D is turned through the angle θ about an axis tangent to D. Assume a fixed value for θ. The turned elements satisfy an equation of the isogonal system and form a surface band intersecting S along D under the angle θ. By the Cauchy existence theorem this band will lie on a unique integral surface of the isogonal system. Then it is clear that any given integral surface of $G = 0$ is cut under a given arbitrary constant angle θ by a one-parameter family of integral surfaces of the isogonal system (21), the curves of intersection on the given surface being the orthogonal trajectories of the characteristics on that surface. This result may be expressed in

THEOREM 2. *Any integral surface of an isogonal system will be cut by a*

one-parameter family of integral surfaces of that system under a given arbitrary constant angle along orthogonal trajectories of the characteristics.

Consider the special case when the characteristics of the original equation $G = 0$ are *lines of curvature on all integral surfaces.* The condition therefor is found by expressing that the direction on an integral surface conjugate to that of the characteristic is at right angles to the characteristic. This conjugate direction (common to all surfaces containing the characteristic*) is given by (7) when δ and ϵ are the denominators of dp and dq, respectively, in (10). In homogeneous coördinates it is readily found that, for this common conjugate direction,

$$\delta x : \delta y : \delta z = p_2 G_z - p_3 G_y : p_3 G_x - p_1 G_z : p_1 G_y - p_2 G_x. \tag{32}$$

The condition sought, namely, $G_1 \delta x + G_2 \delta y + G_3 \delta z = 0$, reduces to

$$\alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z = 0. \tag{33}$$

But each term in (33) is unchanged by T_θ. Hence follows

THEOREM 3. *In the isogonal system $G' = 0$ if the characteristics are lines of curvature on all integral surfaces for any value of θ, then this is true for all values.*

The class of partial differential equations of the first order for which the characteristics are lines of curvature on all integral surfaces forms therefore an invariant system under T_θ, and has been studied from this point of view by Liebmann.† Theorem 1 takes a special form in this case (as proved by Liebmann), since the first osculating circles are now great circles on the principal spheres of one system. The paper of Liebmann deals with these spheres only.

In a plane isogonal system the integral curves may be arranged in isogonal nets, or two one-parameter systems such that each curve of one system will intersect all those of the other under the same angle θ. We inquire if corresponding double families of surfaces exist among the integral surfaces of the isogonal system $G' = 0$. Let S be an integral surface of $G = 0$. For convenience, denote the family (Theorem 2) cutting S by S'_θ. Consider those characteristics on the surfaces S'_θ which intersect one of the characteristics on S. The locus of these ∞^1 characteristics is a surface orthogonal to S, and hence arises a system of surfaces T orthogonal to S intersecting it along the characteristics, and also containing the characteristics on the S'_θ. If now S belongs to a one-parameter family S_0 of integral surfaces of $G = 0$ each of which stands in the same relation to the system S'_θ as S, then the T and S'_θ surfaces must intersect everywhere orthogonally. Furthermore, this

* See p. 524.

† *Aequitangential- und Isogonaltransformationen der partiellen Differentialgleichungen D_{12}*, Rendiconti del Circolo Matematico di Palermo, vol. 29 (1910), p. 139.

fact is to be independent of θ. Consider, in particular, $\theta = 90°$. The T, the S'_θ, and the S_0 surfaces now form a triple orthogonal system and hence intersect along lines of curvature. That is, a *necessary condition* is that the characteristics shall be lines of curvature on the S_0 surfaces. But in this case the characteristics are lines of curvature on the S'_θ for all θ (Theorem 3). Since the T and S'_θ surfaces for every θ are orthogonal, it appears that the characteristics on the S'_θ must be lines of curvature on the T surfaces *for every* θ, that is, the T surfaces must be plane or spherical. A necessary condition for the aforesaid arrangement is thus readily seen to be that all integral surfaces shall be developable or canal surfaces. The isogonal net of plane curves is, therefore, not duplicated for the case under discussion in any general sense.

Certain points in the preceding argument have excluded the linear equation

$$Xp + Yq - Z = 0, \tag{34}$$

in which X, Y, Z are functions of x, y, z, only. The isogonal system arising here is

$$Xp + Yq - Z + \sin\theta \sqrt{1 + p^2 + q^2}\,\sqrt{X^2 + Y^2 + Z^2} = 0, \tag{35}$$

in which the primes on p and q have been dropped. The elementary cones in (35) for a given value of θ are congruent cones of revolution whose axes are the tangents of the characteristics of (34). Equation (35) may be written in the alternative form

$$\left\| \begin{matrix} X & Y & Z \\ p & q & -1 \end{matrix} \right\|^2 = \cos^2\theta, \tag{36}$$

the left-hand member being the sum of the squares of the three determinants of the matrix. For $\theta = 90°$, real integral surfaces exist only if the characteristics of (34) form a normal congruence, and in that case, there are a simple infinity of such surfaces only.

3. Theorems for an equitangential system

In the differential equation

$$F(x, y, z, p, q) = 0 \tag{9}$$

we may inquire as to the locus of the points of the surfaces elements having a common plane

$$z = px + qy + c. \tag{37}$$

This locus is plainly the curve* arising by regarding (9) and (37) simultaneous equations with p and q constant. We may call this curve the *ele-*

* Caratheodory (l. c.) calls this curve "die Elementplankurve."

mentary curve of the plane. What is the direction of the tangent of this curve? The infinitesimals dx, dy, and dz at a point on the curve satisfy

$$F_x\,dx + F_y\,dy + F_z\,dz = 0, \qquad dz = p\,dx + q\,dy, \tag{38}$$

from which the ratios $dx : dy : dz$ are found to be

$$\frac{dx}{-F_y - qF_z} = \frac{dy}{F_x + pF_z} = \frac{dz}{qF_x - pF_y}. \tag{39}$$

This direction in the element (x, y, z, p, q) is precisely the second of the two directions referred to in § 1, namely, *that direction on all integral surfaces containing the element which is conjugate to the direction of the characteristic,*— the *common conjugate direction,* as we shall call it.

It is well now to place in juxtaposition the following statements which appear dual.

(*a*) The planes of the elements satisfying $F = 0$ and having a common point envelop a cone, *the elementary cone of the point.* The elements of the cone are tangent to the characteristics through the point.

(*b*) The points of the elements satisfying $F = 0$ and having a common plane lie on a curve, *the elementary curve of the plane.* The tangents of this curve are the common conjugate directions in the plane.

These facts justify us in assigning to the tangents of the elementary curve of a plane a rôle in the geometry of partial differential equations equal in importance to that of the tangents of the characteristics. This we do in the following developments.

Let (c_1, c_2, c_3) be the direction cosines of the common conjugate direction. Then by (32),

$$c_1 : c_2 : c_3 = \begin{vmatrix} p_1 & p_2 & p_3 \\ G_x & G_y & G_z \end{vmatrix}, \tag{40}$$

the notation meaning that the c's are proportional to the corresponding second order determinants of the matrix of the right-hand member.

In § 2 we defined a turn of a surface element of a partial differential equation of the first order. We now define a slide.

DEFINITION. *A slide of a surface element satisfying a partial differential equation of the first order is a translation through a given distance in its plane in a direction perpendicular to the common conjugate direction.*

Applied to a given equation

$$G(x, y, z, p_1, p_2, p_3) = 0,$$

a slide S_a through the distance a transforms the elements into those satisfying a new equation

$$G'(x', y', z', p_1, p_2, p_3) = 0. \tag{41}$$

The transformed elementary curve in any plane is obviously a *parallel curve* of the original elementary curve. In fact, it is precisely this phase of the matter which suggested the direction of the slide. An immediate result is expressed in

THEOREM 4. *Under a slide the common conjugate direction in an element remains parallel to itself.*

If $(\sigma_1, \sigma_2, \sigma_3)$ are the direction cosines of the line along which the point of the element moves, then the equations of the slide S_a are

$$x' = x + a\sigma_1, \qquad y' = y + a\sigma_2, \qquad z' = z + a\sigma_3. \tag{42}$$

For a given equation $G = 0$, the slides S_a form a group.

It is our purpose to prove a theorem for space analogous to Scheffers' result already quoted for plane equitangential systems. To do this a second osculating circle must be associated with a given pair of united surface elements. Consider a developable surface upon which these elements lie. The direction of the generator of this developable is the common conjugate direction (c). From equations (3), § 1, we find the relation*

$$\alpha\beta(rt - s^2) = \delta\epsilon - (\alpha\delta + \beta\epsilon)s. \tag{43}$$

For a developable surface, $rt - s^2 = 0$, or†

$$s = \frac{\delta\epsilon}{\alpha\beta + \delta\epsilon}. \tag{44}$$

Putting this value of s in equation (5), we find for the curvature of normal sections of the developable,

$$\frac{1}{R_2} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{(\delta\,\delta x + \epsilon\,\delta y)^2}{(\alpha\delta + \beta\epsilon)\,\delta s^2}. \tag{45}$$

Apply this to the direction (σ). For this purpose we merely have to set $\sigma_1 = \delta x/\delta s$, $\sigma_2 = \delta y/\delta s$. Replacing in (45) the values of α, β, δ, ϵ, etc., in homogeneous coördinates as in § 1, simple reductions lead to the result

$$\frac{1}{R_2} = \frac{(\sigma_1 G_x + \sigma_2 G_y + \sigma_3 G_z)^2}{\omega(G_1 G_x + G_2 G_y + G_3 G_z)}, \tag{46}$$

giving the radius R_2 of the *second osculating circle.*

It will be well to describe this *second osculating circle.* The integral surfaces along a characteristic have in common a *characteristic surface band* (in Lie's terminology). This characteristic band will lie on a developable, the *characteristic developable.* The second osculating circle for an element $(x, y, z,$

* See the paper by the author already cited.

† We assume $\alpha\beta + \delta\epsilon \neq 0$, the meaning of which is that the characteristic and common conjugate directions in the element (x, y, z, p, q) are distinct.

p, q) of the characteristic band is the circle osculating the normal section of the characteristic developable which is at right angles to the generator. The first osculating circle is, moreover, the osculating circle of this developable in a normal section along the characteristic. In fact (45) reduces to (8) when $\delta x : \delta y = \alpha : \beta$. The two osculating circles are therefore associated with this developable. Moreover, *when the characteristics are lines of curvature, the two osculating circles are identical.* For the generator of the developable is now at right angles to the characteristic. If ϕ is the angle between the characteristic and common conjugate directions, then, by Euler's Theorem, $R_2 = R_1 \sin^2 \phi$.

Return now to an element (x, y, z, p_1, p_2, p_3) of the differential equation $G = 0$. Consider the second osculating circles of the ∞^1 transformed elements under S_a. These circles lie in a plane (perpendicular to the conjugate direction (c)), and touch the line which is the path of the point of the transformed element. But these ∞^1 circles also *touch a second line,* as we now prove.

Under the transformation

$$x' = x + a\sigma_1, \qquad y' = y + a\sigma_2, \qquad z' = z + a\sigma_3, \tag{42}$$

with $p_1 : p_2 : p_3$ remaining unchanged, the given equation $G = 0$ is transformed into the system

$$G'(x', y', z', p_1, p_2, p_3) = 0. \tag{41}$$

For this equation the curvature of the second osculating circle is, by (46),

$$\frac{1}{R_2'} = \frac{(\sigma_1 G_{x'}' + \sigma_2 G_{y'}' + \sigma_3 G_{z'}')^2}{\omega(G_1' G_{x'}' + G_2' G_{y'}' + G_3' G_{z'}')}. \tag{46'}$$

The partial derivatives of $G = 0$ and $G' = 0$ under (42) are related as follows:

$$G_x = G_{x'}' + a\left(G_{x'}'\frac{\partial\sigma_1}{\partial x} + G_{y'}'\frac{\partial\sigma_2}{\partial x} + G_{z'}'\frac{\partial\sigma_3}{\partial x}\right). \tag{47}$$

The directions (c), (σ), and (λ) (equation (17)), are mutually at right angles. Assume

$$\begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +1. \tag{48}$$

Then the application of (15) gives

$$\frac{\partial\sigma_i}{\partial x} = -c_i \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ c_1 & c_2 & c_3 \\ \dfrac{\partial c_1}{\partial x} & \dfrac{\partial c_2}{\partial x} & \dfrac{\partial c_3}{\partial x} \end{vmatrix} \qquad (i = 1, 2, 3). \tag{49}$$

Denoting the determinant in the right-hand member by Δ_x, equation (47)

becomes

$$G_x = G'_{x'} - a\Delta_x (G'_{x'} c_1 + G'_{y'} c_2 + G'_{z'} c_3). \tag{50}$$

Similar equations hold for G_y and G_z.

Now $c_1 G_x + c_2 G_y + c_3 G_z = 0$ by (40). Hence from (50) will follow the equation

$$(G'_{x'} c_1 + G_{y'} c_2 + G_{z'} c_3)(1 - a(c_1 \Delta_x + c_2 \Delta_y + c_3 \Delta_z)) = 0, \tag{51}$$

which must hold for *all values of* a. In consequence, the first factor must be zero, and hence G_x, G_y, and G_z *remain unaltered by a slide.* The same fact was observed for the turn S_θ (equations (29)). Next,

$$G_1 = G'_1 + a\left(G'_{x'}\frac{\partial \sigma_1}{\partial p_1} + G'_{y'}\frac{\partial \sigma_2}{\partial p_1} + G'_{z'}\frac{\partial \sigma_3}{\partial p_1}\right). \tag{52}$$

Referring again to (15), we may write

$$\frac{\partial \sigma_i}{\partial p_1} = A\lambda_i + Bc_i, \tag{53}$$

in which A and B are determinants. Moreover, $A = -\sigma_1/\omega$, when equations (17) are used. We thus find the final form

$$G'_1 = G_1 + \frac{\sigma_1}{\omega^2}(p_1 G_x)\, a, \tag{54}$$

and similar equations for G'_2 and G'_3, the same abbreviation $(p_1 G_x)$ being used as in (31).

Substitution in (46′) gives

$$\frac{1}{R'_2} = \frac{(\sigma_1 G_x)^2}{\omega(G_1 G_x) + a(p_1 G_x)(\sigma_1 G_x)/\omega}, \tag{55}$$

the parentheses standing for sums of three terms as before. But this equation shows that the centers of the second osculating circles *lie on a line.* For the coördinates of their centers referred to the line (σ) and the diameter of the circle for $a = 0$ as axes, are R'_2 and a, and, in these coördinates, (55) is the equation of a straight line. Hence

THEOREM 5. *When a surface element of a partial differential equation of the first order is slid in a direction at right angles to the common conjugate direction in the element the center of the second osculating circle describes a line.*

The system of transformed equations

$$G'(x', y', z', p_1, p_2, p_3) = 0 \tag{41}$$

may aptly be called an *equitangential system.* For let S be an integral surface for $a = 0$. Under S_a each surface element of S is displaced a distance a in a

direction at right angles to the common conjugate direction. Consider therefore on S the system of curves (Σ) which are the orthogonal trajectories of the system conjugate to the characteristics.* Then each element of S slides along a tangent of a (Σ) curve. Draw the system on S conjugate to the (Σ) curves, and consider one of these curves,—call it γ. Circumscribe to S along γ the tangent developable Γ. The surface elements of S along γ become under S_a for any arbitrary given a a surface band along a curve γ_a upon the tangent developable Γ such that the corresponding surface elements of S and of this band have a common distance of length a. This band along γ_a determines a unique integral surface S_a of the system $G' = 0$. This surface and S have a common tangent developable and the distance on any generator of this developable between the points of contact with S and S_a is constant and equal to a. That is, S and S_a are *equitangential surfaces.* Accordingly we have

THEOREM 6. *Associated with any integral surface in an equitangential system is a one-parameter family of integral surfaces each of which is an equitangential surface of the original surface. The curves of contact of the common tangent developables are conjugate to the orthogonal trajectories of the system conjugate to the characteristics.*

In the preceding section it appeared that the partial differential equations of the first order for which the characteristics are lines of curvature on all integral surfaces form an invariant class under T_θ. A corresponding theorem is true when turns are replaced by slides. For the condition (33) applied to (41) is

$$\alpha_1' G_x + \alpha_2' G_y + \alpha_3' G_z = 0.$$

Now $\alpha_1' = l_2' \lambda_3' - l_3' \lambda_2' = l_2' \lambda_3 - l_3' \lambda_2$, etc., and the l_i' are proportional to the G_i' of equation (54), say $l_i' = \rho G_i'$. Then

$$\alpha_1' = \rho (G_2 \lambda_3 - G_3 \lambda_2) + \frac{a\rho}{\omega^2} (p_1 G_x)(\sigma_2 \lambda_3 - \sigma_3 \lambda_2).$$

Hence

$$\alpha_1' G_x + \alpha_2' G_y + \alpha_3' G_z = \rho' (\alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z), \tag{56}$$

where $\rho' = \sqrt{G_1^2 + G_2^2 + G_3^2}/\sqrt{G_1'^2 + G_2'^2 + G_3'^2}$. We may therefore state a companion theorem to Theorem 3 as

THEOREM 6. *In the equitangential system $G' = 0$ if the characteristics are lines of curvature on all integral surfaces for any value of a, then this is true for all values of a.*

This result was established by Liebmann in the paper cited above.

From formula (55) it appears that R_2' vanishes for one value of a, that is,

* The point is here that the system conjugate to the characteristics takes the rôle assumed by the characteristics in § 2.

$(G_1' G_x) = 0$ for the corresponding surface element. In deriving (45) exception was made of the case $\alpha\delta + \beta\epsilon = 0$, namely, when the common conjugate direction and the characteristic direction coincide. This exceptional case occurs, however, normally for an element of the transformed system, the vanishing of $(G_1' G_x)$ being precisely this case. To include all cases, therefore, we may assume in (46) that $(G_1 G_x)$ may vanish, and no flaw in the argument developed will be occasioned thereby.

4. General theorems

The result established in § 2 as stated in Theorem 1 suggests an investigation along the following lines.

In the differential equation

$$G(x, y, z, p_1, p_2, p_3) = 0, \tag{12}$$

the first osculating circles for an element (x, y, z, p_1, p_2, p_3) and the elements derived by turning it form a coaxal system passing through a common second point $\overline{P}(\bar{x}, \bar{y}, \bar{z})$. Are these circles also first osculating circles at $\overline{P}$ for a second isogonal system of differential equations? This query is the more natural since Scheffers has shown, in the memoir cited above, that the corresponding theorem holds for a plane isogonal system.

Begin by determining the direction of the axis of the coaxal system. Let C be the angle between this axis and the normal to the surface element $E : (x, y, z, p_1, p_2, p_3)$ at P. Obviously C is the value of θ in (31) when R_1' becomes infinite, and hence

$$\tan C = \frac{\omega(l_1 G_x)}{(p_1 G_x)}. \tag{57}$$

For convenience, denote the first osculating circle for the element E by c_0. The radius R_1 of this circle is given by

$$\frac{1}{R_1} = \frac{(l_1 G_x)}{\omega(G)}.$$

Since the circle c_0 is to be the first osculating circle at $\overline{P}$ for an element $\overline{E} : (\bar{x}, \bar{y}, \bar{z}, \bar{p}_1, \bar{p}_2, \bar{p}_3)$ of an equation

$$\overline{G}(\bar{x}, \bar{y}, \bar{z}, \bar{p}_1, \bar{p}_2, \bar{p}_3) = 0, \tag{58}$$

simple considerations show that the relations between the coördinates of E and $\overline{E}$ are

$$\bar{p}_i = p_i \cos 2C + \omega l_i \sin 2C \qquad (i = 1, 2, 3),$$

$$\bar{x} = x + \frac{R_1}{\omega}(\bar{p}_1 - p_1), \qquad \bar{y} = y + \frac{R_1}{\omega}(\bar{p}_2 - p_2), \tag{59}$$

$$\bar{z} = z + \frac{R_1}{\omega}(\bar{p}_3 - p_3).$$

In other words, equation (57) results from the given equation $G = 0$ by the transformation (59).

Express now the partial derivatives of the original equation in terms of those of the transformed equation (58). Omitting details, the relations are as follows.

$$G_x = \bar{G}_x + \frac{1}{\omega} R_{1x} H + 2C_x (I_1 g_1) - \sin 2C (p_1 \alpha_2 \alpha_{3x}) K, \tag{60}$$

and similar equations for G_y and G_z resulting when the subscript x has been replaced by y, etc. Moreover,

$$H = (\bar{p}_1 \bar{G}_{\bar{x}}) - (p_1 \bar{G}_{\bar{x}}) = 2 \sin C (\omega \cos C (l_1 \bar{G}_{\bar{x}}) - \sin C (p_1 \bar{G}_{\bar{x}})),$$

$$I_1 = \frac{R_1}{\omega} \bar{G}_{\bar{x}} + \bar{G}_1, \qquad I_2 = \frac{R_1}{\omega} \bar{G}_{\bar{y}} + \bar{G}_2, \qquad I_3 = \frac{R_1}{\omega} \bar{G}_{\bar{z}} + \bar{G}_3, \tag{61}$$

$$g_i = - p_i \sin 2C + \omega l_i \cos 2C \qquad (i = 1, 2, 3),$$

$$K = (\alpha_1 I_1) = \frac{R_1}{\omega} (\alpha_1 \bar{G}_{\bar{x}}) + (\alpha_1 \bar{G}_1),$$

and $(p_1 \alpha_2 \alpha_{3x})$ is a determinant with the usual notation. The partial derivatives with respect to p_1, p_2, and p_3 are expressed by equations of which the following is typical.

$$G_1 = \bar{G}_1 + \left(\frac{R_0}{\omega}\right)_1 H + 2C_1 (I_1 g_1) - 2 \sin^2 C I_1 + \sin 2C \left(\frac{p_1}{\omega} (l_1 I_1) - \frac{l_1}{\omega} (p_1 I_1) - (p_1 \alpha_2 \alpha_{31}) K \right). \tag{62}$$

Consider now the *necessary* conditions attached to the assumption that c_0 is the first osculating circle of the element $\bar{E}$ of the differential equation $\bar{G} = 0$. Using, as above, a superior bar on symbols associated with $\bar{G} = 0$, then, since the characteristic in $\bar{E}$ must be tangent to c_0 at $\bar{P}$, we must have

$$\frac{\bar{G}_i}{(\bar{G})} = \bar{l}_i = \frac{p_i}{\omega} \sin 2C - l_i \cos 2C = - g_i/\omega \qquad (i = 1, 2, 3), \tag{63}$$

in which $(\bar{G}) = + \sqrt{\bar{G}_1^2 + \bar{G}_2^2 + \bar{G}_3^2}$. Then from (61),

$$(I_1 g_1) = \frac{-\omega}{(\bar{G})} (I_1 \bar{G}_1) = \frac{-\omega}{(\bar{G})} \left[\frac{R_1}{\omega} (\bar{G}_1 \bar{G}_{\bar{x}}) + (\bar{G}_1^2) \right] = - \frac{\omega (\bar{G}_1 \bar{G}_{\bar{x}})}{(\bar{G})} \left(\frac{R_1}{\omega} - \frac{\bar{R}_1}{\omega} \right). \tag{64}$$

Hence the necessary condition $(I_1 g_1) = 0$, since R_1 must equal $\bar{R}_1$.

Recognizing next that the angle C must be the same whether associated with $G = 0$, or $\overline{G} = 0$, we have, comparing with (57),

$$\tan \overline{C} = \frac{\overline{\omega}(\overline{l}_1 \overline{G}_{\overline{x}})}{(\overline{p}_1 \overline{G}_{\overline{x}})} = \tan C.$$

This condition leads easily to $H = 0$. To complete the discussion, multiply equation (62) by α_1, and form the sum $(\alpha_1 G_1)$. Using the conditions $H = (I_1 g_1) = 0$ already found, and remembering that $(\alpha_1 G_1) = (\alpha_1 \overline{G}_1) = 0$, we find that $K = 0$. Then from equations (61) we see that $(\alpha_1 \overline{G}_{\overline{x}}) = 0$, and since, from (59), $G_x = \overline{G}_{\overline{x}}$, also $(\alpha_1 G_x) = 0$. But this condition holds when and only when the characteristics are lines of curvature on all integral surfaces, as shown in (33), Conversely, assuming $(\alpha_1 G_x) = 0$, we may readily show that $(\alpha_1 \overline{G}_1) = H = (I_1 g_1) = K = 0$, and hence this is a sufficient condition. The details in this demonstration may be omitted since the sufficiency condition is implicity demonstrated in the paper of Liebmann referred to above. Thus we have

THEOREM 7. *The first osculating circles for a pencil of corresponding surface elements of an isogonal system $G' = 0$ have the same relation to a second isogonal system at their second common point of intersection when and only when the characteristics are lines of curvature on all integral surfaces in both systems.*

For an equitangential system the discussion proceeds along similar lines with a similar result. Recalling that the configuration arising by continuously sliding an element is a flat straight surface band, we may state

THEOREM 8. *The second osculating circles for a flat straight band of corresponding elements of an equitangential system have the same relation to a second equitangential system along their second common tangent when and only when the characteristics are lines of curvature on all integral surfaces in both systems.*

When an element of $G = 0$ is turned, a relation between the radii of the first and second osculating circles may be derived in the form

$$\frac{1}{R_2} = \frac{1}{R_1} + lR_1, \qquad l = \frac{(\alpha_1 G_x)^2}{\omega^2 (G_1^2)}, \tag{65}$$

and l is an invariant of the turn T_θ. Similarly, when the element is slid,

$$R_1 = R_2 + \frac{L}{R_2}, \qquad L = \frac{(\alpha_1 G_x)^2 \omega^2 (G_1^2)}{(\sigma_1 G_x)^4}, \tag{66}$$

and L is invariant under S_a. The conclusion may be drawn at once that the center of the second osculating circle describes a conic when the element is turned, a similar property holding for the center of the first osculating circle when the element is slid.

In the preceding discussion the derivatives G_x, G_y, and G_z remained un-

changed under both transformations T_θ and S_a. The interpretation of this fact possesses geometric interest. When an element is turned, the common conjugate direction moves in a plane whose normal has the direction established by the ratios $G_x : G_y : G_z$. When an element is transformed by S_a, the line of centers of the second osculating circles has this direction.

One final detail may be mentioned. When an element is slid, the characteristic direction turns about a fixed point in the plane of the element. That is, the tangents to the characteristics form a pencil. Denote the vertex of this pencil by V. Direct the attention upon an element at the point P. When this element is turned, the line of centers of the first osculating circles will pierce the plane of the element in a point W. It may be shown without difficulty that P is the mid-point of V and W.

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ERRATA, VOLUME 18

Page 73. T. H. HILDEBRANDT. *On a theory of linear differential equations in general analysis.*

Page 79, line 25, the expression

$$\sum_{k=1}^{n} c_{ik}\, y_{0kj}(x)$$

should be replaced by

$$\sum_{k=1}^{n} y_{0ik}(x)\, c_{kj}.$$

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